

Asymptotics in shallow water waves

Robert McOwen & Peter Topalov

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Abstract

In this paper we consider the initial value problem for a family of shallow water equations on the line \mathbb{R} with various asymptotic conditions at infinity. In particular we construct solutions with prescribed asymptotic expansion as $x \rightarrow \pm\infty$ and prove their invariance with respect to the solution map.

1 Introduction

In this paper we study the initial value problem for a family of shallow water equations on the line \mathbb{R} that contains the Camassa-Holm equation (CH),

$$\begin{cases} m_t + um_x + 2mu_x = 0, & m := u - u_{xx} \\ u|_{t=0} = u_0 \end{cases} \quad (1)$$

with initial data u_0 that has an asymptotic expansion of order $N \in \mathbb{Z}_{\geq 0}$,

$$u_0(x) = \begin{cases} c_0^+ + \frac{c_1^+}{x} + \dots + \frac{c_N^+}{x^N} + o(|x|^{-N}) & \text{as } x \rightarrow +\infty \\ c_0^- + \frac{c_1^-}{x} + \dots + \frac{c_N^-}{x^N} + o(|x|^{-N}) & \text{as } x \rightarrow -\infty, \end{cases} \quad (2)$$

where c_k^\pm are real constants for $0 \leq k \leq N$. (Note that the asymptotic expansion of u_0 as $x \rightarrow +\infty$ is *not* necessarily the same as the one when $x \rightarrow -\infty$.) We are interested to study whether (1) possesses a solution with the prescribed initial data and if the corresponding solution map (provided it exists) preserves the asymptotic expansion (2) in the sense described below. We postpone for the moment the questions related to the regularity of the initial data u_0 and the solutions.

Note that similar questions were studied for the Korteweg-de Vries equation (KdV) in a series of papers [2, 3, 4] as well as in [16] for the modified KdV equation. In the present paper we develop a different approach to this problem. The new approach is influenced by the seminal paper of Arnold [1] (cf. also the related papers [11, 21, 19, 6]) and is based on the introduction of a *group of asymptotic diffeomorphisms* on the line. The group of asymptotic diffeomorphisms is an infinite dimensional topological group modeled on a Banach space of functions on the line with prescribed asymptotic expansions at infinity. This

provides a general framework for studying the asymptotics of solutions for a relatively large class of nonlinear equations. As a model we consider equation (1) (as well as its generalization (3) below). The approach is *not* restricted to equations with one spacial dimension (cf. [18]).

Since the time when equation (1) was derived (in [12] using algebraic principles and in [5] as a model for shallow water waves), it has been attracting a lot of attention. One of the reasons is that it is completely integrable and, unlike the classical Korteweg-de Vries equation, admits non-smooth solitary traveling waves $u(x, t) = ce^{-|x-ct|}$ called *peakons*.¹ Equation (1) is a particular case of a 1-parameter family of shallow water equations ([13]),

$$\begin{cases} m_t + um_x + bmu_x = 0, & m = u - u_{xx} \\ u|_{t=0} = u_0 \end{cases} \quad (3)$$

where b is a real parameter. When $b = 2$ we get (1); the case $b = 3$ is the Degasperis-Procesi equation (DP). Both equations are known to be completely integrable and to admit solitary traveling waves. It is worth noting that, although the cases when $b \neq 2, 3$ in (3) are not known to be completely integrable, they still admit solitary traveling waves ([14]).

The main result of this paper states that, under some technical assumptions on the remainder term $o(|x|^{-N})$ in (2), for any $b \in \mathbb{R}$ and for any $m \geq 3$ and $N \geq 0$ there exists $T > 0$ and a solution $u \in C^0([0, T], H_{loc}^m(\mathbb{R})) \cap C^1([0, T], H_{loc}^{m-1}(\mathbb{R}))$ of (3) so that for any $t \in [0, T]$, $u(t)$ has the asymptotic expansion (2) with coefficients c_k^\pm for $0 \leq k \leq N$ that may depend on $t \in [0, T]$ – see Theorem 1.1 and Theorem 1.2 below for the precise statement of the result. Here $H_{loc}^m(\mathbb{R})$ denotes the space of measurable real valued functions on \mathbb{R} whose weak derivatives up to order m are locally square integrable.

Analytic set-up: In order to perform analysis on functions satisfying (2) we assume that the remainder term $o(|x|^{-N})$ lies in a suitable weighted Sobolev space. The main idea is to incorporate the asymptotic terms in (2) into a new functional space that we call an *asymptotic space*. It turns out that the asymptotic spaces defined this way are Banach spaces that enjoy the Banach algebra property in the sense that the (pointwise) product of functions is continuous. Another important feature of these spaces is that they admit a natural (graded) Lie algebra structure that corresponds to the group structure of a special class of diffeomorphisms of \mathbb{R} (see below). In this paper we restrict our attention to the following asymptotic spaces:

(1) *The asymptotic space $\mathcal{A}_{n,N}^m$.* Take $m \geq 1$ and $N \geq 0$. In order to define the asymptotic space $\mathcal{A}_{n,N}^m(\mathbb{R})$, we first define the weighted Sobolev space

$$W_N^m(\mathbb{R}) := \{f \in H_{loc}^m(\mathbb{R}) \mid \langle x \rangle^N f, \langle x \rangle^{N+1} f', \dots, \langle x \rangle^{N+m} f^{(m)} \in L^2(\mathbb{R})\}, \quad (4)$$

¹Note that the x -derivative of $u(t, x)$ has a jump of magnitude $2c$ at $x = ct$.

supplied with the norm

$$\|f\|_{W_N^m} := \left(\sum_{j=0}^m \int_{\mathbb{R}} |\langle x \rangle^{N+j} f^{(j)}(x)|^2 dx \right)^{1/2},$$

where $\langle x \rangle := \sqrt{1+x^2}$ and $f^{(j)} = \partial_x^j f$ denotes the j -th weak derivative of f . Note that $f \in W_N^m(\mathbb{R})$ implies (Lemma 6.3)

$$f(x) = o(|x|^{-N}) \text{ as } x \rightarrow \pm\infty. \quad (5)$$

One easily sees that (2) is equivalent to

$$u_0(x) = \sum_{k=0}^N \left(a_k \frac{1}{\langle x \rangle^k} + b_k \frac{x}{\langle x \rangle^{k+1}} \right) + o(\langle x \rangle^{-N}) \quad (6)$$

where the constants $a_k, b_k \in \mathbb{R}$, $0 \leq k \leq m$, are uniquely determined from c_k^\pm , $0 \leq k \leq N$. In view of this we define the asymptotic space

$$\begin{aligned} \mathcal{A}_N^m(\mathbb{R}) &:= \left\{ u = \sum_{k=0}^N \left(a_k \frac{1}{\langle x \rangle^k} + b_k \frac{x}{\langle x \rangle^{k+1}} \right) + f \mid f \in W_N^m(\mathbb{R}) \right\}, \\ \|u\|_{\mathcal{A}_N^m} &:= \sum_{k=0}^N (|a_k| + |b_k|) + \|f\|_{W_N^m}. \end{aligned}$$

By (5), the elements of $\mathcal{A}_N^m(\mathbb{R})$ satisfy the asymptotic expansion (6) (and equivalently, (2)). More generally, for $0 \leq n \leq N$, define

$$\begin{aligned} \mathcal{A}_{n,N}^m(\mathbb{R}) &:= \left\{ u = \sum_{k=n}^N \left(a_k \frac{1}{\langle x \rangle^k} + b_k \frac{x}{\langle x \rangle^{k+1}} \right) + f \mid f \in W_N^m(\mathbb{R}) \right\}, \\ \|u\|_{\mathcal{A}_{n,N}^m} &:= \sum_{k=n}^N (|a_k| + |b_k|) + \|f\|_{W_N^m}. \end{aligned}$$

We set,

$$\mathcal{A}_{n,N}^m(\mathbb{R}) := W_N^m(\mathbb{R}) \text{ for } n \geq N+1. \quad (7)$$

For any $R > 0$ denote by $B_{\mathcal{A}_{n,N}^m}(R)$ the ball of radius R in $\mathcal{A}_{n,N}^m$ centered at the origin. The following theorem is proved in Section 3.

Theorem 1.1. *For any $b \in \mathbb{R}$, $m \geq 3$, $N \geq 0$, $n \geq 1$ and $R > 0$, there exists $T > 0$ such that for any $u_0 \in B_{\mathcal{A}_{n,N}^m}(R)$ there exists a unique solution $u \in C^0([0, T], \mathcal{A}_{n,N}^m(\mathbb{R})) \cap C^1([0, T], \mathcal{A}_{n,N}^{m-1}(\mathbb{R}))$ of (3) that depends continuously on the initial data $u_0 \in B_{\mathcal{A}_{n,N}^m}(R)$ in the sense that the data-to-solution map,*

$$u_0 \mapsto u, \quad B_{\mathcal{A}_{n,N}^m}(R) \rightarrow C^0([0, T], \mathcal{A}_{n,N}^m(\mathbb{R})) \cap C^1([0, T], \mathcal{A}_{n,N}^{m-1}(\mathbb{R})),$$

is continuous. Moreover, the coefficients a_k, b_k , $n \leq k \leq \min\{2n, N\}$, in the asymptotic expansion of the solution $u(t)$ are independent of t .

(2) *The asymptotic space $\mathbb{A}_{n,N}^m$.* Although the choice of the remainder space W_N^m is natural as it mimics the properties of the asymptotic terms $1/\langle x \rangle^k$ and $x/\langle x \rangle^{k+1}$ with respect to the differentiation in the x -direction, this choice is not unique: one can choose other Banach spaces of functions that satisfy the *remainder condition* (5). For any $m \geq 1$ and $N \geq 0$ consider the weighted Sobolev space,

$$H_N^m(\mathbb{R}) := \{f \in H_{loc}^m(\mathbb{R}) \mid \langle x \rangle^N f, \langle x \rangle^N f', \dots, \langle x \rangle^N f^{(m)} \in L^2(\mathbb{R})\}, \quad (8)$$

$$\|f\|_{H_N^m} := \left(\sum_{j=0}^m \int_{\mathbb{R}} |\langle x \rangle^N f^{(j)}(x)|^2 dx \right)^{1/2}.$$

One has the following continuous inclusions: $W_N^m(\mathbb{R}) \subseteq H_N^m(\mathbb{R}) \subseteq H^m(\mathbb{R})$, and, for $1 \leq l \leq N$, $H_N^l(\mathbb{R}) \subseteq W_{N-l}^l(\mathbb{R})$. Moreover, the elements of $H_N^m(\mathbb{R})$ satisfy the remainder condition (5) (see Lemma 6.4). Hence, for $0 \leq n \leq N$, we can define in a similar way as above the asymptotic space,

$$\mathbb{A}_{n,N}^m(\mathbb{R}) := \left\{ u = \sum_{k=n}^N \left(a_k \frac{1}{\langle x \rangle^k} + b_k \frac{x}{\langle x \rangle^{k+1}} \right) + f \mid f \in H_N^m(\mathbb{R}) \right\}, \quad (9)$$

$$\|u\|_{\mathbb{A}_{n,N}^m} := \sum_{k=n}^N (|a_k| + |b_k|) + \|f\|_{H_N^m}.$$

We set,

$$\mathbb{A}_{n,N}^m(\mathbb{R}) := H_N^m(\mathbb{R}) \text{ for } n \geq N+1. \quad (10)$$

For any $R > 0$ denote by $B_{\mathbb{A}_{n,N}^m}(R)$ the ball of radius R in $\mathbb{A}_{n,N}^m$ centered at the origin. The following theorem is proved in Section 4.

Theorem 1.2. *For any $b \in \mathbb{R}$, $m \geq 3$, $N \geq 0$, $n \geq 0$ and $R > 0$, there exists $T > 0$ such that for any $u_0 \in B_{\mathbb{A}_{n,N}^m}(R)$, there exists a unique solution $u \in C^0([0, T], \mathbb{A}_{n,N}^m(\mathbb{R})) \cap C^1([0, T], \mathbb{A}_{n,N}^{m-1}(\mathbb{R}))$ of (3) that depends continuously on the initial data $u_0 \in B_{\mathbb{A}_{n,N}^m}(R)$ in the sense that the data-to-solution map,*

$$u_0 \mapsto u, \quad B_{\mathbb{A}_{n,N}^m}(R) \rightarrow C^0([0, T], \mathbb{A}_{n,N}^m(\mathbb{R})) \cap C^1([0, T], \mathbb{A}_{n,N}^{m-1}(\mathbb{R})),$$

is continuous. Moreover, the coefficients a_k, b_k , $n \leq k \leq \min\{2n, N\}$, in the asymptotic expansion of the solution $u(t)$ are independent of t .

Remark 1.1. *Note that, unlike in Theorem 1.1, the case $n = 0$ is not excluded in Theorem 1.2. Hence, the class of solutions obtained in Theorem 1.2 contains, in particular, solutions $u(t)$ such that for any $t \in [0, T]$, $u(t, x) \rightarrow c_0^\pm$ as $x \rightarrow \pm\infty$. The constants c_0^\pm are not necessarily zero, or equal to each other. This implies that such solutions are not necessarily summable or square integrable.*

Remark 1.2. In view of Theorem 1.1 and Theorem 1.2 the coefficients c_k^\pm , $n \leq k \leq n_* := \max\{2n, N\}$ in the asymptotic expansion of the solution u are conservation laws of equation (3) for any $b \in \mathbb{R}$. Hence we get $2(n_* - n + 1)$ functionally independent integrals of motion. This is in contrast to the KdV and the modified KdV equation where only the leading coefficients c_n^\pm are preserved ([16]).

Remark 1.3. By taking $n = N + 1$ in Theorem 1.1 and Theorem 1.2 we obtain, in view of conventions (7) and (10), that (3) is well-posed in the remainder spaces W_N^m and H_N^m ($m \geq 3$, $N \geq 0$) respectively. The case of the space W_1^3 was considered by Constantin in [6] while the case of $H^3 = H_0^3$ was treated by Constantin-Escher in [7].

Remark 1.4. As the goal of this work is to study the spacial asymptotics of the solutions we do not attempt to lower the regularity exponent m in our spaces. Note however that the method allows for analogs of Theorem 1.1 and Theorem 1.2 to be proved with $m > 3/2$. We refer to [9, 10, 20] where low-regularity results are proved for the CH equation. In order to keep the paper as non-technical as possible, we also do not consider the case of the asymptotic L^p -spaces $\mathcal{A}_{n,N}^{m,p}$ and $\mathbb{A}_{n,N}^{m,p}$, $p \geq 1$ (see [18] for the definitions and the main properties of these spaces). Since we always assume $p = 2$, we simplify the notation by omitting reference to p .

Remark 1.5. The definition of the asymptotic group in Section 2 can be extended by allowing linear terms $c_{-1}^\pm x$, $c_{-1}^\pm \neq -1$ in the asymptotic expansion (2) and the corresponding asymptotic spaces. Note that similarly to KdV, the CH equation (1) possesses the unbounded solution $u(t, x) := x/[3(t - 1)]$ that, at $t = 1$, blows-up at all points $x \in \mathbb{R}$ except $x = 0$.

Organization of the paper: The paper is organized as follows. In Section 2 we define the group of asymptotic diffeomorphisms on the line and discuss its main properties. At the end of the section we formulate Proposition 2.1 that establishes a relation between the solutions of (3) and the solutions of a dynamical system formulated in terms of the asymptotic group. Section 3 is devoted to the proof of Theorem 1.1. Theorem 1.2 is proved in Section 4. Proposition 2.1 is proved in Appendix A. In Appendix B we study the properties of the operator $1 - \partial_x^2$ acting on weighted spaces and, for the convenience of the reader, prove in much detail several auxiliary results needed in the main body of the paper.

2 Groups of asymptotic diffeomorphisms

Denote by $\text{Diff}_+^1(\mathbb{R})$ the group of orientation preserving C^1 -diffeomorphisms on the line \mathbb{R} . For any $m \geq 2$, $N \geq 0$, and $n \geq 0$ define

$$\mathcal{AD}_{n,N}^m := \{\varphi \in \text{Diff}_+^1(\mathbb{R}) \mid \varphi(x) = x + u(x), u \in \mathcal{A}_{n,N}^m\}$$

and

$$\mathbb{AD}_{n,N}^m := \{\varphi \in \text{Diff}_+^1(\mathbb{R}) \mid \varphi(x) = x + u(x), u \in \mathbb{A}_{n,N}^m\}.$$

The topology on $\mathcal{AD}_{n,N}^m$ (resp. $\mathbb{AD}_{n,N}^m$) is inherited in a natural way from the Banach structure of $\mathcal{A}_{n,N}^m$ (resp. $\mathbb{A}_{n,N}^m$). The following theorem follows from the analysis in [18].

Theorem 2.1. 1) For any $m \geq 2$, $N \geq 0$, $n \geq 0$, the composition of maps

$$\circ : \mathcal{A}_{n,N}^m \times \mathcal{AD}_{n,N}^m \rightarrow \mathcal{A}_{n,N}^m, (v, \varphi) \mapsto v \circ \varphi \quad \text{is continuous} \quad (11)$$

and

$$\circ : \mathcal{A}_{n,N}^{m+1} \times \mathcal{AD}_{n,N}^m \rightarrow \mathcal{A}_{n,N}^m, (v, \varphi) \mapsto v \circ \varphi \quad \text{is } C^1\text{-smooth.}$$

2) For any $m \geq 2$, $N \geq 0$, $n \geq 0$, mapping φ to its inverse φ^{-1}

$$\text{Inv} : \mathcal{AD}_{n,N}^{m+1} \rightarrow \mathcal{AD}_{n,N}^{m+1}, \varphi \mapsto \varphi^{-1} \quad \text{is continuous}$$

and

$$\text{Inv} : \mathcal{AD}_{n,N}^{m+1} \rightarrow \mathcal{AD}_{n,N}^m, \varphi \mapsto \varphi^{-1} \quad \text{is } C^1\text{-smooth.}$$

3) The same statements in 1) and 2) are true if \mathcal{A} is replaced by \mathbb{A} .

Corollary 2.1. For $m \geq 3$, $N \geq 0$, $n \geq 0$, both $\mathcal{AD}_{n,N}^m$ and $\mathbb{AD}_{n,N}^m$ are topological groups with respect to composition.

We call $\mathcal{AD}_{n,N}^m$ and $\mathbb{AD}_{n,N}^m$ groups of asymptotic diffeomorphisms. Note that

$$\mathcal{AD}_{n,N}^m \subseteq \mathbb{AD}_{n,N}^m \subseteq \text{Diff}_+^1(\mathbb{R}).$$

Let us also mention another group of diffeomorphisms that was studied in [15]:

$$\mathcal{D}^m(\mathbb{R}) := \{\varphi \in \text{Diff}_+^1(\mathbb{R}) \mid \varphi(x) = x + u(x), u \in H^m\}.$$

Observe that the inclusion $\mathbb{AD}_{n,N}^m \subseteq \mathcal{D}^m(\mathbb{R})$ holds for $n \geq 1$ but $\mathcal{D}^m(\mathbb{R}) \not\subseteq \mathbb{AD}_{0,0}^m$; in fact, by convention (7), $\mathcal{D}^m(\mathbb{R}) = \mathbb{AD}_{1,0}^m$.

Remark 2.1. For simplicity we have omitted \mathbb{R} in the notation for $\mathcal{AD}_{n,N}^m$ and $\mathbb{AD}_{n,N}^m$. Henceforth in this paper we shall also omit \mathbb{R} in the notation for other function spaces.

The significance of these groups of asymptotic diffeomorphisms stems from their use for introducing Lagrangian coordinates. This is summarized in the following two results (in which we use $\dot{\cdot}$ to denote the t -derivative).

Lemma 2.1. Let $m \geq 3$, $N \geq 0$, $n \geq 0$, and $T > 0$. Assume that $u \in C^0([0, T], \mathcal{A}_{n,N}^m)$. Then there exists a unique $\varphi \in C^1([0, T], \mathcal{AD}_{n,N}^m)$ so that

$$\dot{\varphi} = u \circ \varphi, \quad \varphi|_{t=0} = \text{id}. \quad (12)$$

The same statement is true if \mathcal{A} is replaced by \mathbb{A} .

Remark 2.2. *This is an important lemma. It shows in particular that although the asymptotic group $\mathcal{AD}_{n,N}^m$ is not a Lie group in classical sense (as the composition (11) is not C^∞ -smooth²) one can still define the Lie-group exponential map,*

$$\exp_{\mathcal{AD}_{n,N}^m} : \mathcal{A}_{n,N}^m \rightarrow \mathcal{AD}_{n,N}^m, \quad u \mapsto \varphi(1),$$

where $\varphi \in C^1([0, \infty), \mathcal{AD}_{n,N}^m)$ is the solution of (12) and $u \in \mathcal{A}_{n,N}^m$ is independent of t . In this sense, the asymptotic space $\mathcal{A}_{n,N}^m$ can be thought as the “Lie algebra” of $\mathcal{AD}_{n,N}^m$.

The following Proposition establishes a relation between the solutions of (3) and the solutions of a dynamical system formulated in terms of the asymptotic group.

Proposition 2.1. *Assume that $m \geq 3$, $N \geq 0$, $n \geq 1$, and $u_0 \in \mathcal{A}_{n,N}^m$. Then there exists a bijective correspondence between solutions of equation (3) in $C^0([0, T], \mathcal{A}_{n,N}^m) \cap C^1([0, T], \mathcal{A}_{n,N}^{m-1})$ and solutions of*

$$\begin{cases} (\dot{\varphi}, \dot{v}) = (v, R_\varphi \circ (1 - \partial_x^2)^{-1} \circ \beta_b \circ R_{\varphi^{-1}}(v)) \\ (\varphi, v)|_{t=0} = (\text{id}, u_0) \end{cases} \quad (13)$$

in $C^1([0, T], \mathcal{AD}_{n,N}^m \times \mathcal{A}_{n,N}^m)$, where $\beta_b(u) = -b u u_x - (3-b) u_x u_{xx}$ and $R_\varphi(u) = u \circ \varphi$. More specifically, if $(\varphi, v) \in C^1([0, T], \mathcal{AD}_{n,N}^m \times \mathcal{A}_{n,N}^m)$ is a solution of (13) then

$$u = v \circ \varphi^{-1}$$

is a solution of (3) in $C^0([0, T], \mathcal{A}_{n,N}^m) \cap C^1([0, T], \mathcal{A}_{n,N}^{m-1})$. The same statements are true if \mathcal{A} is replaced by \mathbb{A} ; we then allow $n \geq 0$.

Lemma 2.1 and Proposition 2.1 are proved in Appendix A. They will be used in Section 3 to prove Theorem 1.1 and in Section 4 to prove Theorem 1.2.

3 Analysis on the asymptotic space $\mathcal{A}_{n,N}^m$

In this section we prove Theorem 1.1. The proof is based on Proposition 2.1 that establishes a correspondence between solutions of (3) and the dynamical system (13) on $\mathcal{AD}_{n,N}^m \times \mathcal{A}_{n,N}^m$. Our first goal is to prove that the right hand side of the first equation in (13) defines a C^∞ -smooth vector field on an open neighborhood of $(\text{id}, 0)$ in $\mathcal{AD}_{n,N}^m \times \mathcal{A}_{n,N}^m$.

3.1 Mapping Properties of $(1 - \partial_x^2)^{-1}$ on $\mathcal{A}_{n,N}^m$

Let $m \geq 1$, $N \geq 0$, and $n \geq 0$. In order to study the mapping properties of the operator $\Lambda := 1 - \partial_x^2$ and its inverse on \mathcal{A}_N^m we introduce the modified weighted Sobolev space

$$\widetilde{W}_N^{m+2} := \{f \in H_{loc}^{m+2} \mid f \in W_N^m, \langle x \rangle^{N+m} f^{(m+1)}, \langle x \rangle^{N+m} f^{(m+2)} \in L^2\}, \quad (14)$$

²This map is *not* even Lipschitz continuous.

$$\|f\|_{\widetilde{W}_N^{m+2}} := \|f\|_{W_N^m} + \|\langle x \rangle^{N+m} f^{(m+1)}\|_{L^2} + \|\langle x \rangle^{N+m} f^{(m+2)}\|_{L^2}.$$

Accordingly, for $n \geq 0$ we define the modified asymptotic space

$$\tilde{\mathcal{A}}_{n,N}^{m+2} := \left\{ u = \sum_{k=n}^N \left(a_k \frac{1}{\langle x \rangle^k} + b_k \frac{x}{\langle x \rangle^{k+1}} \right) + f \mid f \in \widetilde{W}_N^{m+2}(\mathbb{R}) \right\},$$

$$\|u\|_{\tilde{\mathcal{A}}_{n,N}^{m+2}} := \sum_{k=n}^N (|a_k| + |b_k|) + \|f\|_{\widetilde{W}_N^{m+2}}.$$

It follows directly from the definition of \widetilde{W}_N^{m+2} and $\tilde{\mathcal{A}}_{n,N}^{m+2}$ that

$$\widetilde{W}_N^{m+2} \subseteq W_N^m, \quad \tilde{\mathcal{A}}_{n,N}^{m+2} \subseteq \mathcal{A}_{n,N}^m \quad (N \geq 0) \quad (15)$$

$$\widetilde{W}_N^{m+2} \subseteq W_{N-1}^{m+1}, \quad \tilde{\mathcal{A}}_{n,N}^{m+2} \subseteq \mathcal{A}_{n,N-1}^{m+1} \quad (N \geq 1) \quad (16)$$

$$\widetilde{W}_N^{m+2} \subseteq W_{N-2}^{m+2}, \quad \tilde{\mathcal{A}}_{n,N}^{m+2} \subseteq \mathcal{A}_{n,N-2}^{m+2} \quad (N \geq 2) \quad (17)$$

where the inclusions are continuous. There is a natural splitting

$$\mathcal{A}_{n,N}^m = A_{n,N} \oplus W_N^m \quad \text{and} \quad \tilde{\mathcal{A}}_{n,N}^{m+2} = A_{n,N} \oplus \widetilde{W}_N^{m+2} \quad (18)$$

where the finite-dimensional space

$$A_{n,N} := \left\{ \sum_{k=n}^N \left(a_k \frac{1}{\langle x \rangle^k} + b_k \frac{x}{\langle x \rangle^{k+1}} \right) \mid a_k, b_k \in \mathbb{R} \right\}$$

is supplied with the norm $\sum_{k=n}^N (|a_k| + |b_k|)$. First, we prove the following Lemma.

Lemma 3.1. *For any $m \geq 1$, $N \geq 0$, and $n \geq 0$, the mapping,*

$$\Lambda : \widetilde{W}_N^{m+2} \rightarrow W_N^m, \quad (19)$$

is a linear isomorphism.

Proof. Take $f \in \widetilde{W}_N^{m+2}$. For any $0 \leq j \leq m$,

$$\langle x \rangle^{N+j} [(1 - \partial_x^2) f^{(j)}] = \langle x \rangle^{N+j} f^{(j)} - \langle x \rangle^{N+j} f^{(j+2)}. \quad (20)$$

By the definition of \widetilde{W}_N^{m+2} , for any $0 \leq j \leq m$,

$$\langle x \rangle^{N+j} f^{(j)} \in L^2, \quad (21)$$

and for any $0 \leq j \leq m-2$, with $m \geq 2$,

$$|\langle x \rangle^{N+j} f^{(j+2)}| \leq \langle x \rangle^{N+j+2} |f^{(j+2)}| \in L^2. \quad (22)$$

In addition, for any $m \geq 1$,

$$|\langle x \rangle^{N+m-1} f^{(m+1)}| \leq \langle x \rangle^{N+m} |f^{(m+1)}| \in L^2 \quad (23)$$

and

$$\langle x \rangle^{N+m} f^{(m+2)} \in L^2. \quad (24)$$

Combining together (20)-(24) we see that $\Lambda(f) \in W_N^m$ and (19) is continuous. As $\Lambda : H^{m+2} \rightarrow H^m$ is a linear isomorphism and as $\widetilde{W}_N^{m+2} \subseteq H^{m+2}$, the mapping (19) is injective.

Next, we prove that (19) is onto. Take an arbitrary $g \in W_N^m \subseteq H^m$. Let $f := Q(g)$ where Q is defined by formula (84). It follows from Lemma 6.1 (iii) that $f = Q(g) \in H^{m+2}$ and

$$(1 - \partial_x^2)f = g. \quad (25)$$

Lemma 6.1 (ii) implies that for any $0 \leq j \leq m$,

$$f^{(j)} = Q(g)^{(j)} = \frac{1}{2}(Q_-(g^{(j)}) + Q_+(g^{(j)})),$$

where the operators Q_\pm are defined in (83). This together with Lemma 6.1 (i) then gives that

$$f \in W_N^m. \quad (26)$$

Further, in view of (84) and (85) we have

$$\begin{aligned} f' &= \frac{1}{2}(Q_+(g)' + Q_-(g)') = \frac{1}{2}((g - Q_+(g)) + (Q_-(g) - g)) \\ &= \frac{1}{2}(Q_-(g) - Q_+(g)). \end{aligned} \quad (27)$$

Hence,

$$f^{(m+1)} = Q_-(g^{(m)}) - Q_+(g^{(m)}).$$

As $\langle x \rangle^{N+m} g^{(m)} \in L^2$ we obtain from Lemma 6.1 (i) that

$$\langle x \rangle^{N+m} f^{(m+1)} \in L^2. \quad (28)$$

It follows from (25) that

$$f^{(m+2)} = (f'')^{(m)} = f^{(m)} - g^{(m)}.$$

As $f, g \in W_N^m$ we get that

$$\langle x \rangle^{N+m} f^{(m+2)} \in L^2. \quad (29)$$

Combining (26), (28), and (29) we conclude that $f \in \widetilde{W}_N^{m+2}$, and therefore (19) is onto. Finally, the Lemma follows from the open mapping theorem. \square

As a corollary of Lemma 3.1 we obtain

Proposition 3.1. *For any $m \geq 1$, $N \geq 0$, and $n \geq 0$, the mapping,*

$$\Lambda : \tilde{\mathcal{A}}_{n,N}^{m+2} \rightarrow \mathcal{A}_{n,N}^m, \quad (30)$$

is a linear isomorphism.

Proof. For the sake of further reference we first record that for any $k \geq 0$,

$$\begin{aligned} \left(\frac{1}{\langle x \rangle^k}\right)' &= -k \frac{x}{\langle x \rangle^{k+2}}, \\ \left(\frac{1}{\langle x \rangle^k}\right)'' &= k(k+1) \frac{1}{\langle x \rangle^{k+2}} - k(k+2) \frac{1}{\langle x \rangle^{k+4}} \end{aligned} \quad (31)$$

and

$$\begin{aligned} \left(\frac{x}{\langle x \rangle^{k+1}}\right)' &= -k \frac{1}{\langle x \rangle^{k+1}} + (k+1) \frac{1}{\langle x \rangle^{k+3}}, \\ \left(\frac{x}{\langle x \rangle^{k+1}}\right)'' &= k(k+1) \frac{x}{\langle x \rangle^{k+3}} - (k+1)(k+3) \frac{x}{\langle x \rangle^{k+5}}. \end{aligned} \quad (32)$$

As a consequence, for any $k \geq N+1$ and for any $r \geq 1$,

$$\frac{1}{\langle x \rangle^k}, \frac{x}{\langle x \rangle^{k+1}} \in \widetilde{W}_N^{r+2} \subseteq W_N^r. \quad (33)$$

This implies that for any $k \geq 0$ and for any $r \geq 1$,

$$\frac{1}{\langle x \rangle^k}, \frac{x}{\langle x \rangle^{k+1}} \in \tilde{\mathcal{A}}_N^{r+2} \subseteq \mathcal{A}_N^{r+2}. \quad (34)$$

Take $u \in \tilde{\mathcal{A}}_{n,N}^{m+2}$,

$$u = \sum_{k=n}^N \left(a_k \frac{1}{\langle x \rangle^k} + b_k \frac{x}{\langle x \rangle^{k+1}} \right) + f, \quad f \in \widetilde{W}_N^{m+2}.$$

In view of the splitting (18), (31), and (32), we have,

$$\Lambda(u) = \Lambda_1(u) \oplus \Lambda_2(u) \quad (35)$$

where

$$\begin{aligned} \Lambda_1(u) &:= \sum_{k=n}^{N-2} \left(a_k \Lambda\left(\frac{1}{\langle x \rangle^k}\right) + b_k \Lambda\left(\frac{x}{\langle x \rangle^{k+1}}\right) \right) \\ &\quad - \sum_{k=N-3}^{N-2} \left(k(k+2) a_k \frac{1}{\langle x \rangle^{k+4}} + (k+1)(k+3) b_k \frac{x}{\langle x \rangle^{k+5}} \right) \\ &\quad + \sum_{k=N-1}^N \left(a_k \frac{1}{\langle x \rangle^k} + b_k \frac{x}{\langle x \rangle^{k+1}} \right) \end{aligned} \quad (36)$$

and

$$\begin{aligned} \Lambda_2(u) &:= - \sum_{k=N-1}^N \left(a_k \frac{1}{\langle x \rangle^k} + b_k \frac{x}{\langle x \rangle^{k+1}} \right)'' \\ &\quad + \sum_{k=N-3}^{N-2} \left(k(k+2) a_k \frac{1}{\langle x \rangle^{k+4}} + (k+1)(k+3) b_k \frac{x}{\langle x \rangle^{k+5}} \right) \\ &\quad + \Lambda(f). \end{aligned} \quad (37)$$

Here and below we use the convention that a sum vanishes if its upper limit is strictly less than its lower one. It follows from (35)-(37) and Lemma 3.1 that (30) is continuous. Moreover, (30) is injective by Lemma 6.1 (iv): by (89), $\Lambda(u) = 0$ implies $u = Q(\Lambda(u)) = 0$.

Next, we prove that (30) is onto. Take an arbitrary $v \in \mathcal{A}_{n,N}^m$,

$$v = \sum_{k=n}^N \left(a_k \frac{1}{\langle x \rangle^k} + b_k \frac{x}{\langle x \rangle^{k+1}} \right) + g, \quad g \in W_N^m. \quad (38)$$

For any $k \geq 0$ and $l \geq 0$ we have

$$\begin{aligned} \frac{1}{\langle x \rangle^k} &= \Lambda\left(\frac{1}{\langle x \rangle^k}\right) + \left(\frac{1}{\langle x \rangle^k}\right)'' = \Lambda\left(\frac{1}{\langle x \rangle^k}\right) + \left(\Lambda\left(\frac{1}{\langle x \rangle^k}\right) + \left(\frac{1}{\langle x \rangle^k}\right)''\right)'' \\ &= \Lambda\left(\frac{1}{\langle x \rangle^k} + \left(\frac{1}{\langle x \rangle^k}\right)''\right) + \left(\frac{1}{\langle x \rangle^k}\right)^{(4)} = \dots \\ &= \Lambda\left(\sum_{j=0}^l \left(\frac{1}{\langle x \rangle^k}\right)^{(2j)}\right) + \left(\frac{1}{\langle x \rangle^k}\right)^{(2l+2)} \end{aligned} \quad (39)$$

and, similarly,

$$\frac{x}{\langle x \rangle^{k+1}} = \Lambda\left(\sum_{j=0}^l \left(\frac{x}{\langle x \rangle^{k+1}}\right)^{(2j)}\right) + \left(\frac{x}{\langle x \rangle^{k+1}}\right)^{(2l+2)}. \quad (40)$$

Taking $l \geq 0$ such that $k + (2l + 2) \geq N + 1$ we get from (31)-(33) that

$$\left(\frac{1}{\langle x \rangle^k}\right)^{(2l+2)}, \left(\frac{x}{\langle x \rangle^{k+1}}\right)^{(2l+2)} \in W_N^m. \quad (41)$$

Hence, in view (39), (40), (41) and Lemma 3.1, and (34), for any $k \geq 0$ there exists $u_k, w_k \in \mathcal{A}_{n,N}^{m+2}$ such that

$$\frac{1}{\langle x \rangle^k} = \Lambda(u_k) \quad \text{and} \quad \frac{x}{\langle x \rangle^{k+1}} = \Lambda(w_k).$$

This together with Lemma 3.1 and (38) implies that there exists $u \in \tilde{\mathcal{A}}_{n,N}^{m+2}$ such that

$$v = \Lambda(u).$$

Finally, the Proposition follows from the open mapping theorem. \square

Combining Proposition 3.1 with (17) we get

Corollary 3.1. *For any $m \geq 3$, $N \geq 0$, and $n \geq 0$, the mapping,*

$$(1 - \partial_x^2)^{-1} : \mathcal{A}_{n,N+2}^{m-2} \rightarrow \mathcal{A}_{n,N}^m,$$

is well-defined and continuous.

3.2 Smoothness of the conjugate map

Assume that $m \geq 3$, $N \geq 0$, and $n \geq 0$. In this section we study the *conjugate map*

$$(\varphi, v) \mapsto (\varphi, R_\varphi \circ (1 - \partial_x^2) \circ R_{\varphi^{-1}}(v)), \quad \mathcal{AD}_{n,N}^m \times \tilde{\mathcal{A}}_{n,N+2}^m \xrightarrow{\sigma} \mathcal{AD}_{n,N}^m \times \mathcal{A}_{n,N+2}^{m-2} \quad (42)$$

where φ^{-1} denotes the inverse element of φ in $\mathcal{AD}_{n,N}^m$, $R_\varphi(v) := v \circ \varphi$, and the symbol \circ denotes the composition of mappings.

Lemma 3.2. *For any $m \geq 3$, $N \geq 0$, and $n \geq 0$, the mapping (42) is well-defined and C^∞ -smooth.*

Proof. Let $(\varphi, v) \in \mathcal{AD}_{n,N}^m \times \tilde{\mathcal{A}}_{n,N+2}^m$. As $m \geq 3$, one gets from Lemma 6.3 that the C^1 -diffeomorphism $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $\varphi'(x) = 1 + o(1)$ as $|x| \rightarrow \infty$. This implies

$$0 < \inf_{x \in \mathbb{R}} \varphi'(x) < \infty. \quad (43)$$

As $(\varphi^{-1})' = 1/\varphi' \circ \varphi^{-1}$ we obtain from Lemma 6.2 that

$$R_\varphi \circ \partial_x \circ R_{\varphi^{-1}}(v) = R_\varphi \left(v' \circ \varphi^{-1} \cdot \frac{1}{\varphi' \circ \varphi^{-1}} \right)$$

and

$$\begin{aligned} R_\varphi \circ \partial_x^2 \circ R_{\varphi^{-1}}(v) &= R_\varphi \left(v'' \circ \varphi^{-1} \cdot \left(\frac{1}{\varphi' \circ \varphi^{-1}} \right)^2 \right. \\ &\quad \left. - v' \circ \varphi^{-1} \cdot \varphi'' \circ \varphi^{-1} \cdot \left(\frac{1}{\varphi' \circ \varphi^{-1}} \right)^3 \right) \\ &= v'' \cdot \left(\frac{1}{\varphi'} \right)^2 - v' \cdot \varphi'' \cdot \left(\frac{1}{\varphi'} \right)^3. \end{aligned}$$

Hence,

$$R_\varphi \circ (1 - \partial_x^2) \circ R_{\varphi^{-1}}(v) = v - v'' \cdot \left(\frac{1}{\varphi'} \right)^2 + v' \cdot \varphi'' \cdot \left(\frac{1}{\varphi'} \right)^3. \quad (44)$$

Now we claim that

$$(\varphi, v) \mapsto v'' \cdot \left(1/\varphi' \right)^2, \quad \mathcal{AD}_{n,N}^m \times \tilde{\mathcal{A}}_{n,N+2}^m \rightarrow \mathcal{A}_{n,N+2}^{m-2}, \quad (45)$$

and

$$(\varphi, v) \mapsto v' \cdot \varphi'' \cdot \left(1/\varphi' \right)^3, \quad \mathcal{AD}_{n,N}^m \times \tilde{\mathcal{A}}_{n,N+2}^m \rightarrow \mathcal{A}_{n,N+2}^{m-2}, \quad (46)$$

are both C^∞ -smooth. In fact, let us first verify that

$$\varphi \mapsto 1/\varphi', \quad \mathcal{AD}_{n,N}^m \rightarrow \mathcal{A}_{0,N+1}^{m-1}, \quad (47)$$

is C^∞ -smooth. This follows from Lemma 6.10 as $\varphi' = 1 + u' > 0$, $u \in \mathcal{A}_{n,N}^m$, and as the linear map $u \mapsto u'$, $\mathcal{A}_{n,N}^m \rightarrow \mathcal{A}_{n+1,N+1}^{m-1}$, is continuous by Lemma

6.6. Note that any continuous linear (or multilinear) map is C^∞ -smooth. In particular, by the continuity of the pointwise product in $\mathcal{A}_{0,N+1}^{m-1}$ we conclude from (47) that

$$\varphi \mapsto \left(1/\varphi'\right)^2, \quad \mathcal{AD}_{n,N}^m \rightarrow \mathcal{A}_{0,N+1}^{m-1}, \quad \text{and} \quad \varphi \mapsto \left(1/\varphi'\right)^3, \quad \mathcal{AD}_{n,N}^m \rightarrow \mathcal{A}_{0,N+1}^{m-1}, \quad (48)$$

are C^∞ -smooth. Now to show that the map (45) is C^∞ -smooth, we use that the inclusion $\tilde{\mathcal{A}}_{n,N+2}^m \subseteq \mathcal{A}_{n,N}^m$ is continuous (see (17)) to conclude that

$$v \mapsto v'', \quad \tilde{\mathcal{A}}_{n,N+2}^m \rightarrow \mathcal{A}_{n+2,N+2}^{m-2} \subseteq \mathcal{A}_{n+1,N+2}^{m-2}, \quad (49)$$

is smooth. Then the smoothness of (45) follows from the smoothness of the maps (48), (49), and the continuity of the pointwise product (Lemma 6.6), $(f, g) \mapsto f \cdot g, \quad \mathcal{A}_{n+1,N+2}^{m-2} \times \mathcal{A}_{0,N+1}^{m-1} \rightarrow \mathcal{A}_{n,N+2}^{m-2}$.

Similarly, we use the boundedness of the inclusion $\tilde{\mathcal{A}}_{n,N+2}^m \subseteq \mathcal{A}_{n,N+1}^{m-1}$, to obtain the smoothness of $v \mapsto v', \quad \tilde{\mathcal{A}}_{n,N+2}^m \rightarrow \mathcal{A}_{n+1,N+2}^{m-2}$, which together with the smoothness of the maps $\varphi \mapsto \varphi'', \quad \mathcal{AD}_{n,N}^m \rightarrow \mathcal{A}_{n+2,N+2}^{m-2} \subseteq \mathcal{A}_{0,N+2}^{m-2}$, and (48), implies that (46) is C^∞ -smooth.

Finally, in view of the continuity of the inclusion $\tilde{\mathcal{A}}_{n,N+2}^m \subseteq \mathcal{A}_{n,N+2}^{m-2}$, one gets from (44), (45), and (46) that (42) is C^∞ -smooth. \square

The main result of this subsection is the following Proposition.

Proposition 3.2. *For any $m \geq 3$, $N \geq 0$, and $n \geq 0$, there exists an open neighborhood \mathcal{U} of the identity id in $\mathcal{AD}_{n,N}^m$ such that the restriction of the map (42) to $\mathcal{U} \times \tilde{\mathcal{A}}_{n,N+2}^m$ is a C^∞ -diffeomorphism, i.e. if $\mathcal{C} := \sigma|_{\mathcal{U} \times \tilde{\mathcal{A}}_{n,N+2}^m}$ then*

$$\mathcal{C} : \mathcal{U} \times \tilde{\mathcal{A}}_{n,N+2}^m \rightarrow \mathcal{U} \times \mathcal{A}_{n,N+2}^{m-2}, \quad (50)$$

is a C^∞ -diffeomorphism.

Proof. By Lemma 3.2 the map (42) is C^∞ -smooth. The differential of σ at the point $(\text{id}, 0)$, $d_{(\text{id}, 0)}\sigma : \mathcal{A}_{n,N}^m \times \tilde{\mathcal{A}}_{n,N+2}^m \rightarrow \mathcal{A}_{n,N}^m \times \mathcal{A}_{n,N+2}^{m-2}$, is given by

$$(\delta\varphi, \delta v) \mapsto (\delta\varphi, (1 - \partial_x^2)\delta v). \quad (51)$$

Then, it follows from Proposition 3.1 that (51) is a linear isomorphism. Hence, by the inverse function theorem in Banach spaces, there exist an open neighborhood \mathcal{U} of the identity id in $\mathcal{AD}_{n,N}^m$ and an open neighborhood \mathcal{V} of the zero in $\tilde{\mathcal{A}}_{n,N+2}^m$ such that

$$\sigma|_{\mathcal{U} \times \mathcal{V}} : \mathcal{U} \times \mathcal{V} \rightarrow \mathcal{U} \times \mathcal{A}_{n,N+2}^{m-2} \quad (52)$$

is a C^∞ -diffeomorphism onto its image. Here we used that $\sigma(\varphi, 0) = (\varphi, 0)$ for any $\varphi \in \mathcal{AD}_{n,N}^m$. Note also that for any $(\varphi, v) \in \mathcal{AD}_{n,N}^m \times \tilde{\mathcal{A}}_{n,N+2}^m$,

$$\pi_1 \circ \sigma(\varphi, v) = \varphi \quad \text{and} \quad \pi_2 \circ \sigma(\varphi, v) = R_\varphi \circ (1 - \partial_x^2) \circ R_{\varphi^{-1}}(v),$$

where $\pi_1 : \mathcal{AD}_{n,N}^m \times \mathcal{A}_{n,N+2}^{m-2} \rightarrow \mathcal{AD}_{n,N}^m$ and $\pi_2 : \mathcal{AD}_{n,N}^m \times \mathcal{A}_{n,N+2}^{m-2} \rightarrow \mathcal{A}_{n,N+2}^{m-2}$ denote the cartesian projections onto the first and the second component. This and the fact that (52) is a C^∞ -diffeomorphism onto its image imply that for any $\varphi \in \mathcal{U}$ the linear mapping $\pi_2 \circ \sigma(\varphi, \cdot) : \mathcal{V} \rightarrow \mathcal{A}_{n,N+2}^{m-2}$ is continuous, injective, and maps \mathcal{V} onto an open neighborhood of zero in $\mathcal{A}_{n,N+2}^{m-2}$. This and the open mapping theorem then imply that

$$\delta v \mapsto R_\varphi \circ (1 - \partial_x^2) \circ R_{\varphi^{-1}}(\delta v), \quad \tilde{\mathcal{A}}_{n,N+2}^m \rightarrow \mathcal{A}_{n,N+2}^{m-2}, \quad (53)$$

is a linear isomorphism of Banach spaces. In particular, the mapping

$$\mathcal{C} = \sigma|_{\mathcal{U} \times \tilde{\mathcal{A}}_{n,N+2}^m} : \mathcal{U} \times \tilde{\mathcal{A}}_{n,N+2}^m \rightarrow \mathcal{U} \times \mathcal{A}_{n,N+2}^{m-2}, \quad (54)$$

is bijective.

Finally, by computing the differential of \mathcal{C} at an arbitrary point $(\varphi, v) \in \mathcal{V} \times \tilde{\mathcal{A}}_{n,N+2}^m$ we obtain that

$$d_{(\varphi,v)}\mathcal{C}(\delta\varphi, \delta v) = \begin{bmatrix} \text{id}_{\mathcal{A}_{n,N}^m} & 0 \\ * & R_\varphi \circ (1 - \partial_x^2) \circ R_{\varphi^{-1}} \end{bmatrix} \begin{bmatrix} \delta\varphi \\ \delta v \end{bmatrix} \quad (55)$$

where $\text{id}_{\mathcal{A}_{n,N}^m} : \mathcal{A}_{n,N}^m \rightarrow \mathcal{A}_{n,N}^m$ is the identity in $\mathcal{A}_{n,N}^m$ and $*$ denotes a bounded linear map $\mathcal{A}_{n,N}^m \times \tilde{\mathcal{A}}_{n,N+2}^m \rightarrow \mathcal{A}_{n,N+2}^{m-2}$. As (53) is a linear isomorphism we conclude from (55) that

$$d_{(\varphi,v)}\mathcal{C} : \mathcal{A}_{n,N}^m \times \tilde{\mathcal{A}}_{n,N+2}^m \rightarrow \mathcal{A}_{n,N}^m \times \mathcal{A}_{n,N+2}^{m-2}$$

is a linear isomorphism. Applying the inverse function theorem we get that \mathcal{C} is a local C^∞ -diffeomorphism. As \mathcal{C} is bijective we conclude that it is a C^∞ -diffeomorphism. \square

3.3 Smoothness of the vector field

Take $b \in \mathbb{R}$, $m \geq 3$, and $N \geq 0$, and $n \geq 0$. Here we consider the mapping,

$$(\varphi, v) \xrightarrow{\mathcal{B}_b} (\varphi, R_\varphi \circ \beta_b \circ R_{\varphi^{-1}}(v)), \quad \mathcal{AD}_{n,N}^m \times \mathcal{A}_{n,N}^m \xrightarrow{\mathcal{B}_b} \mathcal{AD}_{n,N}^m \times \mathcal{A}_{n,N+2}^{m-2}, \quad (56)$$

where

$$\beta_b(u) := -b u u_x - (3 - b) u_x u_{xx}.$$

First we prove the following Proposition.

Proposition 3.3. *For any $n \geq 1$ the mapping (56) is well-defined and C^∞ -smooth.*

Proof. We follow the lines of the proof of Lemma 3.2. For any $(\varphi, v) \in \mathcal{AD}_{n,N}^m \times \mathcal{A}_{n,N}^m$ one has in view of Lemma 6.2,

$$\begin{aligned} \mathcal{B}_b(\varphi, v) &= R_\varphi \circ \left(-bv \circ \varphi^{-1} \cdot (v \circ \varphi^{-1})' - (3 - b) (v \circ \varphi^{-1})' \cdot (v \circ \varphi^{-1})'' \right) \\ &= -b v \cdot v' \cdot \frac{1}{\varphi'} + (3 - b) (v')^2 \cdot \varphi'' \cdot \left(\frac{1}{\varphi'} \right)^4 - (3 - b) v' \cdot v'' \cdot \left(\frac{1}{\varphi'} \right)^3. \end{aligned} \quad (57)$$

Using Lemma 6.6 and Lemma 6.10 we get,

$$v \in \mathcal{A}_{n,N}^m, \quad \frac{1}{\varphi'} \in \mathcal{A}_{0,N+1}^{m-1}, \quad \text{and} \quad v' \in \mathcal{A}_{n+1,N+1}^{m-1}.$$

Hence, in view of Lemma 6.6, $v \cdot \frac{1}{\varphi'} \in \mathcal{A}_{n,N}^{m-1}$, and therefore,

$$v' \cdot \left(v \cdot \frac{1}{\varphi'} \right) \in \mathcal{A}_{2n+1,N+n+1}^{m-1} \subseteq \mathcal{A}_{n,N+2}^{m-2},$$

where we used that $n \geq 1$. This combined with the continuity of the pointwise product (Lemma 6.6) implies that

$$(\varphi, v) \mapsto v' \cdot \left(v \cdot \frac{1}{\varphi'} \right), \quad \mathcal{AD}_{n,N}^m \times \mathcal{A}_{n,N}^m \rightarrow \mathcal{A}_{n,N+2}^{m-2}$$

is C^∞ -smooth. The other terms in (57) are treated similarly. \square

Remark 3.1. *The proof of Proposition 3.3 shows that*

$$\mathcal{B}_b(\mathcal{AD}_{n,N}^m \times \mathcal{A}_{n,N}^m) \subseteq \mathcal{AD}_{n,N}^m \times \mathcal{A}_{2n+1,N+2}^{m-2}.$$

Finally, combining Proposition 3.2, Proposition 3.3, and the fact that the inclusion,

$$\tilde{\mathcal{A}}_{2n+1,N+2}^m \subseteq \mathcal{A}_{2n+1,N}^m,$$

is continuous, we see that for $n \geq 1$ the mapping,

$$\mathcal{C}^{-1} \circ \mathcal{B}_b|_{\mathcal{U} \times \mathcal{A}_{n,N}^m} : \mathcal{U} \times \mathcal{A}_{n,N}^m \rightarrow \mathcal{U} \times \mathcal{A}_{2n+1,N}^m \subseteq \mathcal{U} \times \mathcal{A}_{n,N}^m, \quad (58)$$

where \mathcal{U} is the open neighborhood given by Proposition 3.2 is C^∞ -smooth. In particular, we see that the mapping,

$$\begin{aligned} (\varphi, v) &\xrightarrow{\mathcal{F}_b} \left(v, R_\varphi \circ (1 - \partial_x^2)^{-1} \circ \beta_b \circ R_{\varphi^{-1}}(v) \right), \\ \mathcal{U} \times \mathcal{A}_{n,N}^m &\xrightarrow{\mathcal{F}_b} \mathcal{A}_{n,N}^m \times \mathcal{A}_{n,N}^m, \end{aligned} \quad (59)$$

is C^∞ -smooth. Hence, we proved the following Theorem.

Theorem 3.1. *For any $m \geq 3$, $N \geq 0$, and $n \geq 1$, the mapping (59) is C^∞ -smooth. In addition, $\mathcal{F}_b(\mathcal{U} \times \mathcal{A}_{n,N}^m) \subseteq \mathcal{A}_{n,N}^m \times \mathcal{A}_{2n+1,N}^m$.*

In view of Theorem 3.1, we see that \mathcal{F}_b defines a C^∞ -smooth vector field on $\mathcal{U} \times \mathcal{A}_{n,N}^m \subseteq \mathcal{AD}_{n,N}^m \times \mathcal{A}_{n,N}^m$.

3.4 Proof of Theorem 1.1

In view of Theorem 3.1 the right-hand side of the first equation in (13) is a C^∞ -smooth vector field on $\mathcal{U} \times \mathcal{A}_{n,N}^m$. Hence, by the existence, uniqueness, and continuous (or smooth) dependence on parameters theorem for ODE's in

Banach spaces [17], there exists $R' > 0$ and $T' > 0$ so that for any u_0 in the ball $B_{\mathcal{A}_{n,N}^m}(R')$ of radius R' in $\mathcal{A}_{n,N}^m$ centered at the origin, there exists a unique solution (φ, v) of (13) in $C^1([0, T'], \mathcal{U} \times \mathcal{A}_{n,N}^m)$. This solution depends continuously on the initial data $u_0 \in B_{\mathcal{A}_{n,N}^m}(R')$, in the sense that the data-to-solution map,

$$u_0 \mapsto (\varphi, v), \quad B_{\mathcal{A}_{n,N}^m}(R') \rightarrow C^1([0, T'], \mathcal{U} \times \mathcal{A}_{n,N}^m),$$

is continuous. In view of Proposition 2.1, the curve $u := v \circ \varphi^{-1}$ is the unique solution of (3) in $C^0([0, T'], \mathcal{A}_{n,N}^m) \cap C^1([0, T'], \mathcal{A}_{n,N}^{m-1})$ and, by item 1) of Theorem 2.1, u depends continuously on the initial data $u_0 \in B_{\mathcal{A}_{n,N}^m}(R')$, in the sense that the data-to-solution map,

$$u_0 \mapsto u, \quad B_{\mathcal{A}_{n,N}^m}(R') \rightarrow C^0([0, T'], \mathcal{A}_{n,N}^m) \cap C^1([0, T'], \mathcal{A}_{n,N}^{m-1}),$$

is continuous. Next, recall that the solutions of (3) possess the following scaling invariance: *if $u(t)$, $t \in [0, \tau]$, $\tau > 0$, is a solution of (3) then for any $\lambda > 0$, $u_\lambda(t) := \lambda u(\lambda t)$, $t \in [0, \tau/\lambda]$, is a solution of the first equation in (3) so that $u|_{t=0} = \lambda u_0$.* Now, take an arbitrary $R > R'$ and denote $T := \mu T'$ where $\mu := R'/R$. Let $w_0 \in B_{\mathcal{A}_{n,N}^m}(R)$. Then, $u_0 := \mu w_0 \in B_{\mathcal{A}_{n,N}^m}(R')$ and hence, there exists a unique solution u of equation (3) in $C^0([0, T'], \mathcal{A}_{n,N}^m) \cap C^1([0, T'], \mathcal{A}_{n,N}^{m-1})$. By the scaling invariance, $w(t) := u(t/\mu)/\mu$ is a solution of the first equation in (3) in $C^0([0, T], \mathcal{A}_{n,N}^m) \cap C^1([0, T], \mathcal{A}_{n,N}^{m-1})$ so that $w|_{t=0} = w_0$. The solution w is necessarily unique. Otherwise, using scaling invariance, we will obtain that u is not unique. The same argument also shows that the data-to-solution map $w_0 \mapsto w$, $B_{\mathcal{A}_{n,N}^m}(R) \rightarrow C^0([0, T], \mathcal{A}_{n,N}^m) \cap C^1([0, T], \mathcal{A}_{n,N}^{m-1})$, is continuous. This completes the proof of the first statement of Theorem 1.1.

The second statement can be proved as follows: If $n \geq N + 1$ then by convention (7) there are no asymptotic terms and the statement trivially holds. Assume that $0 \leq n \leq N$ and let $(\varphi, v) \in C^1([0, T], \mathcal{AD}_{n,N}^m \times \mathcal{A}_{n,N}^m)$ be the solution of (13). Then, by Proposition 2.1,

$$u := v \circ \varphi^{-1}$$

is the solution of (3) in $C^0([0, T], \mathcal{A}_{n,N}^m) \cap C^1([0, T], \mathcal{A}_{n,N}^{m-1})$. In view of the last statement of Theorem 3.1, the coefficients,

$$a_k, b_k, \quad n \leq k \leq n_* := \min\{2n, N\}, \quad (60)$$

in the asymptotic expansion of v are independent of $t \in [0, T]$. Then, by Lemma 6.3, for any $t \in [0, T]$,

$$v(t) = \sum_{k=n}^{n_*} \left(a_k \frac{1}{\langle x \rangle^k} + b_k \frac{x}{\langle x \rangle^{k+1}} \right) + o\left(\frac{1}{\langle x \rangle^{n_*}} \right). \quad (61)$$

By Lemma 6.3 and Lemma 6.11, for any $t \in [0, T]$ and for any $n \leq k \leq n_*$,

$$\frac{1}{\langle \cdot \rangle} \circ \varphi = \frac{1}{\langle x \rangle^k} + O\left(\frac{1}{\langle x \rangle^{n+1+k}} \right) = \frac{1}{\langle x \rangle^k} + O\left(\frac{1}{\langle x \rangle^{2n+1}} \right). \quad (62)$$

This and Lemma 6.3 also imply that

$$\begin{aligned} \frac{x}{\langle x \rangle^{k+1}} \circ \varphi &= \left(x + O\left(\frac{1}{\langle x \rangle^n}\right) \right) \left(\frac{1}{\langle x \rangle^{k+1}} + O\left(\frac{1}{\langle x \rangle^{n+2+k}}\right) \right) \\ &= \frac{x}{\langle x \rangle^{k+1}} + O\left(\frac{1}{\langle x \rangle^{2n+1}}\right) \end{aligned} \quad (63)$$

Hence, in view of (61), (62), and (63), for any $t \in [0, T]$,

$$u = v \circ \varphi^{-1} = \sum_{k=n}^{n_*} \left(a_k \frac{1}{\langle x \rangle^k} + b_k \frac{x}{\langle x \rangle^{k+1}} \right) + o\left(\frac{1}{\langle x \rangle^{n_*}}\right).$$

This shows that the first $2(n_* - n + 1)$ coefficients in the asymptotic expansion of u coincide with (60) and therefore they are independent of $t \in [0, T]$.

4 Analysis on the asymptotic space $\mathbb{A}_{n,N}^m$

In this section we prove Theorem 1.2.

4.1 Mapping Properties of $(1 - \partial_x^2)^{-1}$ on $\mathbb{A}_{n,N}^m$

Take $m \geq 1$, $N \geq 0$, and $n \geq 0$. The analogs of Lemma 3.1 and Proposition 3.1 are easier to state and prove since we do not have to modify the spaces H_N^{m+2} and \mathbb{A}_N^{m+2} in order to describe the mapping properties of $\Lambda = 1 - \partial_x^2$.

Lemma 4.1. *For any $m \geq 1$, $N \geq 0$, and $n \geq 0$, the mapping*

$$\Lambda : H_N^{m+2} \rightarrow H_N^m \quad (64)$$

is a linear isomorphism.

Proof. Clearly (64) is continuous, so we only need to verify it is injective and surjective. Similar to the proof of Lemma 3.1, injectivity follows from $H_N^{m+2} \subset H^{m+2}$ and the isomorphism $\Lambda : H^{m+2} \rightarrow H^m$, while the surjectivity follows from Lemma 6.1 (i). \square

Proposition 4.1. *For any $m \geq 1$, $N \geq 0$, and $n \geq 0$, the mapping*

$$\Lambda : \mathbb{A}_{n,N}^{m+2} \rightarrow \mathbb{A}_{n,N}^m \quad (65)$$

is a linear isomorphism.

Proof. The continuity of (65) is clear from the splitting (35)-(37) and Lemma 4.1 and the injectivity follows from Lemma 6.1 (iv). The surjectivity follows by replacing W_N^m by H_N^m in (38) - (41) and using Lemma 4.1. \square

Remark 4.1. *By allowing $m = 0$ in the definition of the asymptotic space $\mathcal{A}_{n,N}^m$ (resp. $\mathbb{A}_{n,N}^m$) and in the corresponding remainder space W_N^m (resp. H_N^m) one can easily verify that the mapping properties in Lemmas 3.1 and 4.1 and in Propositions 3.1 and 4.1 are true for $m = 0$. (In fact, since $\widetilde{W}_N^2 = H_N^2$, both results 3.1 and 4.1 coincide.) However, the reason that we have specified $m \geq 1$ is that it suffices for our main purpose – the proof of Theorems 1.1 and 1.2.*

4.2 Smoothness of the conjugate map

Assume that $m \geq 3$ and consider the conjugate map

$$(\varphi, v) \xrightarrow{\sigma} (\varphi, R_\varphi \circ (1 - \partial_x^2) \circ R_{\varphi^{-1}}(v)), \quad \mathbb{AD}_{n,N}^m \times \mathbb{A}_{n,N}^m \xrightarrow{\sigma} \mathbb{AD}_{n,N}^m \times \mathbb{A}_{n,N}^{m-2} \quad (66)$$

where φ^{-1} denotes the inverse of φ in $\mathbb{AD}_{n,N}^m$. Arguing as in Section 3.2 one gets from Lemma 6.7, Lemma 6.8, and Lemma 6.10, the following Lemma.

Lemma 4.2. *For any $m \geq 3$, $N \geq 0$, and $n \geq 0$, the mapping (66) is well-defined and C^∞ -smooth.*

As a consequence, we prove as in Section 3.2,

Proposition 4.2. *For any $m \geq 3$, $N \geq 0$, and $n \geq 0$, there exists an open neighborhood \mathcal{U} of the identity id in $\mathbb{AD}_{n,N}^m$ such that the restriction of the map (66) to $\mathcal{U} \times \mathbb{A}_{n,N}^m$ is C^∞ -smooth, i.e. if $\tilde{C} := \sigma|_{\mathcal{U} \times \mathbb{A}_{n,N}^m}$ then*

$$C : \mathcal{U} \times \mathbb{A}_{n,N}^m \rightarrow \mathcal{U} \times \mathbb{A}_{n,N}^{m-2} \quad (67)$$

is a C^∞ -diffeomorphism.

4.3 Smoothness of the vector field

Take $b \in \mathbb{R}$ and consider the mapping,

$$(\varphi, v) \xrightarrow{B_b} (\varphi, R_\varphi \circ \beta_b \circ R_{\varphi^{-1}}), \quad \mathbb{AD}_{n,N}^m \times \mathbb{A}_{n,N}^m \xrightarrow{B_b} \mathbb{AD}_{n,N}^m \times \mathbb{A}_{n,N}^{m-2}, \quad (68)$$

where $\beta_b(u) = -b u u_x - (3 - b) u_x u_{xx}$. As in Section 3.3 one has

Proposition 4.3. *For any $m \geq 3$, $N \geq 0$, and $n \geq 0$, the mapping (68) is well-defined and C^∞ -smooth.*

Remark 4.2. *Note that, unlike in Proposition 3.3, the case $n = 0$ is not excluded in Proposition 4.3.*

Proof. As in the proof of Proposition 3.3 we see that for any $(\varphi, v) \in \mathbb{AD}_{n,N}^m \times \mathbb{A}_{n,N}^m$,

$$\begin{aligned} B_b(\varphi, v) &= R_\varphi \circ \left(-b v \circ \varphi^{-1} \cdot (v \circ \varphi^{-1})' - (3 - b) (v \circ \varphi^{-1})' \cdot (v \circ \varphi^{-1})'' \right) \\ &= -b v \cdot v' \cdot \frac{1}{\varphi'} + (3 - b) (v')^2 \cdot \varphi'' \cdot \left(\frac{1}{\varphi'} \right)^4 - (3 - b) v' \cdot v'' \cdot \left(\frac{1}{\varphi'} \right)^3. \end{aligned} \quad (69)$$

By Lemma 6.8 and Lemma 6.10,

$$v \in \mathbb{A}_{n,N}^m, \quad v' \in \mathbb{A}_{n+1,N}^{m-1}, \quad \text{and} \quad \frac{1}{\varphi'} \in \mathbb{A}_{0,N}^{m-1}.$$

In view of Lemma 6.8, $v \cdot \frac{1}{\varphi'} \in \mathbb{A}_{n,N}^{m-1}$, and hence,

$$v' \cdot \left(v \cdot \frac{1}{\varphi'} \right) \in \mathbb{A}_{2n+1,N+d}^{m-1} \subseteq \mathbb{A}_{n,N}^{m-2},$$

This combined with the continuity of the pointwise product (Lemma 6.8) implies that

$$(\varphi, v) \mapsto v' \cdot \left(v \cdot \frac{1}{\varphi'} \right), \quad \mathbb{A}\mathcal{D}_{n,N}^m \times \mathbb{A}_{n,N}^m \rightarrow \mathbb{A}_{2n+1,N}^{m-2} \subseteq \mathbb{A}_{2n+1,N}^{m-2} \subseteq \mathbb{A}_{n,N}^{m-2} \quad (70)$$

is C^∞ -smooth. The other terms in (69) are treated similarly. In fact, as

$$v' \in \mathbb{A}_{n+1,N}^{m-1}, \quad v'' \in \mathbb{A}_{n+2,N}^{m-2}, \quad \varphi'' \in \mathbb{A}_{n+2,N}^{m-2}, \quad \text{and} \quad \frac{1}{\varphi'} \in \mathbb{A}_{0,N}^{m-1},$$

we see from the continuity of the pointwise product (Lemma 6.8) that

$$(\varphi, v) \mapsto (v')^2 \cdot \varphi'' \cdot \left(\frac{1}{\varphi'} \right)^4, \quad \mathbb{A}\mathcal{D}_{n,N}^m \times \mathbb{A}_{n,N}^m \rightarrow \mathbb{A}_{2n+3,N}^{m-2} \subseteq \mathbb{A}_{n,N}^{m-2} \quad (71)$$

and

$$(\varphi, v) \mapsto v' \cdot v'' \cdot \left(\frac{1}{\varphi'} \right)^3, \quad \mathbb{A}\mathcal{D}_{n,N}^m \times \mathbb{A}_{n,N}^m \rightarrow \mathbb{A}_{2n+3,N}^{m-2} \subseteq \mathbb{A}_{n,N}^{m-2} \quad (72)$$

are C^∞ -smooth. In conclusion, we obtain from (69)-(72) that

$$(\varphi, v) \mapsto B_b(\varphi, v), \quad \mathbb{A}\mathcal{D}_{n,N}^m \times \mathbb{A}_{n,N}^m \rightarrow \mathbb{A}_{2n+1,N}^{m-2} \subseteq \mathbb{A}_{n,N}^{m-2} \quad (73)$$

is C^∞ -smooth. \square

Now, arguing as in Section 3.2 we see that the mapping,

$$\begin{aligned} (\varphi, v) &\xrightarrow{F_b} \left(v, R_\varphi \circ (1 - \partial_x^2)^{-1} \circ \beta_b \circ R_{\varphi^{-1}}(v) \right), \\ \mathcal{U} \times \mathbb{A}_{n,N}^m &\xrightarrow{F_b} \mathbb{A}_{n,N}^m \times \mathbb{A}_{2n+1,N}^m \subseteq \mathbb{A}_{n,N}^m \times \mathbb{A}_{n,N}^m, \end{aligned} \quad (74)$$

where \mathcal{U} is the open neighborhood given by Proposition 4.2 is C^∞ -smooth. Hence, we proved the following Theorem.

Theorem 4.1. *For any $m \geq 3$, $N \geq 0$, and $n \geq 0$, the mapping (74) is C^∞ -smooth. In addition, $F_b(\mathcal{U} \times \mathbb{A}_{n,N}^m) \subseteq \mathbb{A}_{n,N}^m \times \mathbb{A}_{2n+1,N}^m$.*

Remark 4.3. *Note that, unlike in Theorem 3.1, the case $n = 0$ is not excluded in Theorem 4.1.*

In view of Theorem 4.1, we see that F_b defines a C^∞ -smooth vector field on $\mathcal{U} \times \mathbb{A}_{n,N}^m \subseteq \mathbb{A}\mathcal{D}_{n,N}^m \times \mathbb{A}_{n,N}^m$.

4.4 Proof of Theorem 1.2

The proof of Theorem 1.2 follows by the same arguments as in the proof of Theorem 1.1.

5 Appendix A: Lagrangian description

In this Appendix we give the proofs of Lemma 2.1 and Proposition 2.1.

Proof of Lemma 2.1. Let $u \in C^0([0, T], \mathcal{A}_{n,N}^m)$. Denote $F(t, \varphi) := u(t) \circ \varphi$. Then by Theorem 2.1,

$$F : [0, T] \times \mathcal{AD}_{n,N}^{m-1} \rightarrow \mathcal{AD}_{n,N}^{m-1}$$

and its partial derivative with respect to the second variable³

$$D_2 F : [0, T] \times \mathcal{AD}_{n,N}^{m-1} \rightarrow \mathcal{L}(\mathcal{A}_{n,N}^{m-1}, \mathcal{A}_{n,N}^{m-1})$$

are continuous. This implies that F is *locally Lipschitz continuous* i.e. for any $(t_0, \varphi_0) \in [0, T] \times \mathcal{AD}_{n,N}^{m-1}$ there exists an open neighborhood V of (t_0, φ_0) in $[0, T] \times \mathcal{AD}_{n,N}^{m-1}$ and $K > 0$ such that for any (t, φ_1) and (t, φ_2) in V ,

$$\|F(t, \varphi_2) - F(t, \varphi_1)\|_{\mathcal{A}_{n,N}^{m-1}} \leq K \|\varphi_2 - \varphi_1\|_{\mathcal{A}_{n,N}^{m-1}}.$$

This together with the existence theorem for ODE's in Banach spaces [17] implies that for any $t_0 \in [0, T]$ there exists an open neighborhood U_{t_0} of (t_0, id) in $[0, T] \times \mathcal{AD}_{n,N}^{m-1}$ and $\varepsilon_{t_0} > 0$ such that for any $(\tau, \psi) \in U_{t_0}$ there exists a unique solution $\varphi \in C^1([0, T] \cap (\tau - \varepsilon_{t_0}, \tau + \varepsilon_{t_0}), \mathcal{AD}_{n,N}^{m-1})$ of $\dot{\varphi} = u \circ \varphi$, $\varphi|_{t=\tau} = \psi$. In view of the compactness of $[0, T] \times \text{id}$ in $[0, T] \times \mathcal{AD}_{n,N}^{m-1}$ we see that there exists $\varepsilon > 0$ such that for any $t_0 \in [0, T]$ there exists a unique solution $\varphi \in C^1([0, T] \cap (t_0 - \varepsilon, t_0 + \varepsilon), \mathcal{AD}_{n,N}^{m-1})$ of

$$\dot{\varphi} = u \circ \varphi, \quad \varphi|_{t=t_0} = \text{id}. \quad (75)$$

Note that if $\varphi \in C^1([0, T] \cap (t_0 - \varepsilon, t_0 + \varepsilon), \mathcal{AD}_{n,N}^{m-1})$ is a solution of (75) then for any $\psi \in \mathcal{AD}_{n,N}^{m-1}$ the curve $t \mapsto \varphi(t) \circ \psi$, $\varphi \circ \psi \in C^1([0, T] \cap (t_0 - \varepsilon, t_0 + \varepsilon), \mathcal{AD}_{n,N}^{m-1})$ is a solution of

$$\dot{\varphi} = u \circ \varphi, \quad \varphi|_{t=t_0} = \psi. \quad (76)$$

This solution is unique as $F(t, \varphi) = u(t) \circ \varphi$ is locally Lipschitz continuous on $[0, T] \times \mathcal{AD}_{n,N}^{m-1}$. As $\varepsilon > 0$ is independent of the choice of $t_0 \in [0, T]$ and $\psi \in \mathcal{AD}_{n,N}^{m-1}$ we conclude that (12) has a unique solution

$$\varphi \in C^1([0, T], \mathcal{AD}_{n,N}^{m-1}).$$

As $\partial_x : \mathcal{A}_{n,N}^{m-1} \rightarrow \mathcal{A}_{n+1,N+1}^{m-2}$ is a bounded linear map we see from Lemma 6.2 (i) that

$$(\varphi_x)^\cdot = u_x \circ \varphi \cdot \varphi_x \quad (77)$$

where the t -derivative is understood in $\mathcal{A}_{n+1,N+1}^{m-2}$. As $\varphi \in C^1([0, T], \mathcal{AD}_{n,N}^{m-1})$ we see that $\varphi_x \in C^1([0, T], \mathcal{AD}_{n+1,N+1}^{m-2})$. By the inclusion $\mathcal{A}_{n+1,N+1}^{m-2} \subseteq L^\infty$ one

³Here $\mathcal{L}(V, W)$ denotes the Banach space of bounded linear maps between two Banach spaces V and W supplied with the uniform operator norm.

concludes that for any given $x \in \mathbb{R}$, $\varphi_x(\cdot, x) \in C^1([0, T], \mathbb{R})$. This together with $\varphi_x(t, x) > 0$ implies that $(\log \varphi_x(t, x))' = u_x(t, \varphi(t, x))$ for any given $x \in \mathbb{R}$ and $t \in [0, T]$. Hence, for any $x \in \mathbb{R}$ and for any $t \in [0, T]$,

$$\begin{aligned}\varphi_x(t, x) &= e^{\int_0^t (u_x(s) \circ \varphi(s))(x) ds} \\ &= 1 + \sum_{k \geq 1} \left(\int_0^t (u_x(s) \circ \varphi(s))(x) ds \right)^k / k!.\end{aligned}\quad (78)$$

Note that by assumption $u_x \in C^0([0, T], \mathcal{A}_{n+1, N+1}^{m-1})$. As $\mathcal{A}_{n+1, N+1}^{m-1} \subseteq \frac{1}{\langle x \rangle} \mathcal{A}_{n, N}^{m-1}$ and as $\varphi \in C^1([0, T], \mathcal{AD}_{n, N}^{m-1})$ we conclude from Theorem 2.1, Lemma 6.11, and Lemma 6.6 that $u_x \circ \varphi \in C^0([0, T], \mathcal{A}_{n+1, N+1}^{m-1})$. This implies that

$$\int_0^t u_x(s) \circ \varphi(s) ds \in \mathcal{A}_{n+1, N+1}^{m-1}$$

as the integrand is a continuous curve in $\mathcal{A}_{n+1, N+1}^{m-1}$. As $\mathcal{A}_{n+1, N+1}^{m-1}$ is a Banach algebra we conclude from (78) that

$$\varphi_x(t) - 1 \in \mathcal{A}_{n+1, N+1}^{m-1}$$

and

$$\varphi_x - 1 \in C^1([0, T], \mathcal{A}_{n+1, N+1}^{m-1}).$$

As in addition $\varphi \in C^1([0, T], \mathcal{AD}_{n, N}^{m-1})$ we see that

$$\varphi \in C^1([0, T], \mathcal{AD}_{n, N}^m).$$

This solution is unique in $\mathcal{AD}_{n, N}^m$ as it is unique as a curve in $\mathcal{AD}_{n, N}^{m-1}$. \square

Proof of Proposition 2.1. Let $u \in C^0([0, T], \mathcal{A}_{n, N}^m) \cap C^1([0, T], \mathcal{A}_{n, N}^{m-1})$ be a solution of (3). Then we have

$$(1 - \partial_x^2)(u_t + uu_x) = -buu_x - (3 - b)u_x u_{xx}.$$

It follows from Lemma 6.6 that $u_t + uu_x \in \mathcal{A}_{n, N}^{m-1}$ and $\beta_b(u) \in \mathcal{A}_{n, N}^{m-2}$. In view of Lemma 6.1 (iv), we get

$$u_t + uu_x = (1 - \partial_x^2)^{-1}(\beta_b(u)) \quad (79)$$

where $(1 - \partial_x^2)^{-1}f := Q(f)$ and $Q(f)$ is defined by (84).

Next, consider the differential equation

$$\dot{\varphi} = u \circ \varphi, \quad \varphi|_{t=0} = \text{id}. \quad (80)$$

In view of Lemma 2.1 there exists a unique solution

$$\varphi \in C^1([0, T], \mathcal{AD}_{n, N}^m).$$

Denote

$$v := u \circ \varphi.$$

It follows from Sobolev embedding theorem that $u(t, x)$ and $\varphi(t, x)$ are functions in $C^1([0, T] \times \mathbb{R}, \mathbb{R})$. In particular, by (79),

$$\begin{aligned} v_t &= u_t \circ \varphi + u_x \circ \varphi \cdot \varphi_t \\ &= (u_t + uu_x) \circ \varphi \\ &= [(1 - \partial_x^2)^{-1}(\beta_b(u))] \circ \varphi. \end{aligned}$$

Hence, for any given $x \in \mathbb{R}$ and $t \in [0, T]$,

$$v(t, x) = u_0(x) + \int_0^t ([(1 - \partial_x^2)^{-1}(\beta_b(u(s)))] \circ \varphi(s))(x) dt. \quad (81)$$

As $u \in C^0([0, T], \mathcal{A}_{n,N}^m)$ we obtain from Lemma 6.6 and $n \geq 1$ that

$$\beta_b(u) \in C^0([0, T], \mathcal{A}_{n,N+2}^{m-2}).$$

In view of Corollary 3.1 we get

$$(1 - \partial_x^2)^{-1}(\beta_b(u)) \in C^0([0, T], \mathcal{A}_{n,N}^m).$$

As $\varphi \in C^1([0, T], \mathcal{AD}_{n,N}^m)$ we obtain from Theorem 2.1 that

$$[(1 - \partial_x^2)^{-1}(\beta_b(u))] \circ \varphi \in C^0([0, T], \mathcal{A}_{n,N}^m).$$

This implies that the integrand in (81) converges in $\mathcal{A}_{n,N}^m$. Hence,

$$v \in C^1([0, T], \mathcal{A}_{n,N}^m)$$

and

$$\dot{v} = R_\varphi \circ (1 - \partial_x^2)^{-1} \circ \beta_b \circ R_{\varphi^{-1}}(v).$$

Conversely, assume that

$$(\varphi, v) \in C^1([0, T], \mathcal{AD}_{n,N}^m \times \mathcal{A}_{n,N}^m)$$

is a solution of (13). Denote

$$u := v \circ \varphi^{-1}.$$

In view of Theorem 2.1,

$$u \in C^0([0, T], \mathcal{A}_{n,N}^m) \cap C^1([0, T], \mathcal{A}_{n,N}^{m-1}).$$

By inspection one checks that u is a solution of (3).

The fact that the described above correspondence between solutions of (3) and (13) is bijective follows from the fact that if $u_1, u_2 \in C^0([0, T], \mathcal{A}_{n,N}^m)$, $u_1 \neq u_2$ then the corresponding solutions $\varphi_1, \varphi_2 \in C^1([0, T], \mathcal{AD}_{n,N}^m)$ of (80) do not coincide. \square

6 Appendix B: Auxiliary Results

In this Appendix we collect for the convenience of the reader some auxiliary results needed in the main body of the paper. For any $\gamma \in \mathbb{R}$ denote

$$L_\gamma^2 := \{f \in L_{loc}^2 \mid \langle x \rangle^\gamma f \in L^2\}.$$

Let

$$L_*^2 := \bigcup_{\gamma \in \mathbb{R}} L_\gamma^2. \quad (82)$$

For any $g \in L_*^2$ define the integral transforms

$$(Q_\pm(g))(x) := \int_0^\infty g(x \mp z) e^{-z} dz \quad (83)$$

and

$$Q(g) := \frac{1}{2}(Q_+(g) + Q_-(g)). \quad (84)$$

Lemma 6.1. (i) For any $g \in L_*^2$, $Q_\pm(g) \in H_{loc}^1$ and

$$(1 \pm \partial_x)Q_\pm(g) = g. \quad (85)$$

Moreover, for any $\gamma \in \mathbb{R}$,

$$\langle x \rangle^\gamma f \in L^2 \implies \langle x \rangle^\gamma Q_\pm(f) \in L^2.$$

(ii) For any $g \in H_{loc}^1 \cap L_*^2$, $Q_\pm(g) \in H_{loc}^2$. If in addition $g' \in L_*^2$ then

$$Q_\pm(g)' = Q_\pm(g') \quad \text{and} \quad (1 \pm \partial_x)Q_\pm(g) = Q_\pm((1 \pm \partial_x)g) = g. \quad (86)$$

The restriction of Q_\pm to H^1 is a linear isomorphism $Q_\pm|_{H^1} : H^1 \rightarrow H^2$.

(iii) For any $g \in H_{loc}^1 \cap L_*^2$, $Q(g) \in H_{loc}^3$, and

$$(1 - \partial_x^2)Q(g) = g. \quad (87)$$

If in addition $g' \in L_*^2$ then

$$Q(g)' = Q(g'). \quad (88)$$

The restriction of Q to H^1 is a linear isomorphism $Q|_{H^1} : H^1 \rightarrow H^3$.

(iv) Assume that $f \in H_{loc}^2$ and $f, f', f'' \in L_*^2$. Then

$$Q((1 - \partial_x^2)f) = (1 - \partial_x^2)Q(f) = f. \quad (89)$$

Proof. (i) Take $g \in L_*^2$. As $g \in L_\gamma^2$ for some $\gamma \in \mathbb{R}$ one sees that for any $a \in \mathbb{R}$ the function $y \mapsto g(\mp y) e^{-y} = (g(\mp y) \langle y \rangle^\gamma) \cdot (e^{-y} \langle y \rangle^{-\gamma})$ is summable on $[a, \infty)$ and

$$(Q_\pm(g))(x) = \int_0^\infty g(x \mp z) e^{-z} dz = e^{\mp x} \int_{\mp x}^\infty g(\mp y) e^{-y} dy. \quad (90)$$

This formula implies that $Q_{\pm}(g)$ is an *absolutely continuous* function on any finite interval of \mathbb{R} and its derivative⁴ belongs to L^2_{loc} . Differentiating (90) we see that,

$$Q_{\pm}(g)' = \pm g \mp Q_{\pm}(g), \quad (91)$$

which implies (85). As $Q_{\pm}(g) \in L^2_{loc}$ we get from (91) that $Q_{\pm}(g) \in H^1_{loc}$.

Now, consider the second statement in (i). Take $\gamma \geq 0$ and note that for any $x, z \in \mathbb{R}$,

$$\langle x \rangle^2 \leq 2\langle x \pm z \rangle^2 \langle z \rangle^2. \quad (92)$$

It follows from (92) and Young's inequality that for any measurable function f such that $\langle x \rangle^\gamma f \in L^2$, $\langle x \rangle^\gamma Q_{\pm}(f) \in L^2$ and

$$\begin{aligned} \|\langle x \rangle^\gamma Q_{\pm}(f)\|_{L^2} &= \left\| \int_0^\infty \langle x \rangle^\gamma f(x-z) e^{-z} dz \right\|_{L^2} \\ &\leq 2^{\gamma/2} \left\| \int_0^\infty \langle x-z \rangle^\gamma |f(x-z)| \langle z \rangle^\gamma e^{-z} dz \right\|_{L^2} \\ &= 2^{\gamma/2} \left\| (\langle \cdot \rangle^\gamma |f(\cdot)|) * (\langle \cdot \rangle^\gamma \chi_{[0,\infty)}(\cdot) e^{-\cdot}) \right\|_{L^2} \leq C \|\langle x \rangle^\gamma f\|_{L^2}, \end{aligned}$$

where $\chi_{[0,\infty)}$ is the characteristic function of $[0, \infty)$ and $C > 0$ is independent of the choice of f . The case of the transformation Q_- is considered in the same way. If $\gamma < 0$ one uses the inequality $\langle x \pm z \rangle^2 \leq 2\langle x \rangle^2 \langle z \rangle^2$ and argues similarly.

(ii) As $g \in H^1_{loc} \cap L^2_*$ we see from (i) that $Q_{\pm}(g) \in H^1_{loc}$ and $Q_{\pm}(g)' = g - Q_{\pm}(g)$. This implies that $Q_{\pm}(g) \in H^2_{loc}$.

Assume in addition that $g' \in L^2_*$. Then, for any test function $\psi \in C_0^\infty$ such that $\text{supp } \psi \subseteq (-R, R)$, $0 < R < \infty$, we get from Fubini's theorem and an integration by parts that,

$$\begin{aligned} \langle Q_{\pm}(g)', \psi \rangle &:= - \int_{-\infty}^\infty \psi'(x) \int_0^\infty g(x \mp z) e^{-z} dz dx \\ &= - \int_0^\infty e^{-z} \int_{-R}^R g(x \mp z) \psi'(x) dx dz \\ &= \int_0^\infty e^{-z} \int_{-R}^R g'(x \mp z) \psi(x) dx dz \\ &= \int_{-\infty}^\infty \left(\int_0^\infty g'(x \mp z) e^{-z} dz \right) \psi(x) dx \\ &= \langle Q_{\pm}(g'), \psi \rangle. \end{aligned} \quad (93)$$

We can apply Fubini's theorem as by the first statement of (i), $\int_0^\infty |g(x \mp z)| e^{-z} dz$ and $\int_0^\infty |g'(x \mp z)| e^{-z} dz$ are locally summable functions in the variable x . Formula (93) then implies that,

$$Q_{\pm}(g)' = Q_{\pm}(g'). \quad (94)$$

⁴Recall that the (pointwise) derivative of an absolutely continuous function exists a.e. and coincides with its weak derivative.

This together with (91) implies that

$$Q_{\pm}((1 \pm \partial_x)g) = (1 \pm \partial_x)Q_{\pm}(g) = g. \quad (95)$$

Now, we prove the last statement in (ii). Assume that $g \in H^1$. Then $g, g' \in L^2 \subseteq L^2_*$, and in view of (94) and the second statement in (i) (with $\alpha = 0$) we see that $Q_{\pm}(g)' = Q_{\pm}(g') \in H^1$. This shows that $Q_{\pm}(g) \in H^2$. Hence, the linear map $Q_{\pm}|_{H^1} : H^1 \rightarrow H^2$ is well defined. It follows from (95) that $Q_{\pm}|_{H^1} : H^1 \rightarrow H^2$ is a bijective map with a continuous inverse $(1 \pm \partial_x) : H^2 \rightarrow H^1$. This together with the open mapping theorem imply that $Q_{\pm}|_{H^1} : H^1 \rightarrow H^2$ is a linear isomorphism.

(iii) This item follows immediately from (i) and (ii). In fact, for any $g \in H^1_{loc} \cap L^2_*$ one gets from (84) and (ii) that $Q(g) \in H^2_{loc}$. Moreover,

$$\begin{aligned} (1 - \partial_x^2)Q(g) &= \frac{1}{2} \left((1 - \partial_x)(1 + \partial_x)Q_+(g) + (1 + \partial_x)(1 - \partial_x)Q_-(g) \right) \\ &= \frac{1}{2} \left((1 - \partial_x)g + (1 + \partial_x)g \right) = g. \end{aligned} \quad (96)$$

This implies that $Q(g)'' = g - Q(g) \in H^1_{loc}$, and therefore $Q(g) \in H^3_{loc}$. Assume that $g \in H^1$. Then it follows from (84) and the last statement in (ii) that $Q(g) \in H^2$. In view of (96), $Q(g)'' = g - Q(g) \in H^1$, and hence, $Q(g) \in H^3$. Hence, the linear map $Q|_{H^1} : H^1 \rightarrow H^3$ is well defined. Now, take $f \in H^3$. Then, by applying (94) twice we get

$$Q((1 - \partial_x^2)f) = (1 - \partial_x^2)Q(f) = f.$$

This implies that $Q|_{H^1} : H^1 \rightarrow H^3$ is a bijective map with a continuous inverse $(1 - \partial_x^2) : H^3 \rightarrow H^1$. Hence, by the open mapping theorem $Q|_{H^1} : H^1 \rightarrow H^3$ is a linear isomorphism.

The proof of (iv) follows from (iii). \square

Lemma 6.2. (i) Assume that $f \in H^1_{loc}$ and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be an orientation preserving C^1 -diffeomorphism of \mathbb{R} . Then, $R_{\varphi}(f) = f \circ \varphi \in H^1_{loc}$ and

$$(f \circ \varphi)' = f' \circ \varphi \cdot \varphi'.$$

If $f \in H^1$ and $0 < \inf_{x \in \mathbb{R}} \varphi'(x) < \infty$ then $f \circ \varphi \in H^1$.

(ii) Assume that $\psi := 1 + f > 0$ where $f \in H^1_{loc}$.⁵ Then $1/\psi \in H^1_{loc}$ and

$$\left(\frac{1}{\psi} \right)' = -\frac{1}{\psi^2} \cdot \psi'.$$

If $f \in H^1$ and $0 < \inf_{x \in \mathbb{R}} \psi(x)$ then $(1/\psi)' \in L^2$.

⁵ $\psi > 0$ means that for any $x \in \mathbb{R}$, $\psi(x) > 0$.

Proof. As the both items of the Lemma follow in a similar way from a simple approximation argument, we prove only (i). Assume that $f \in H_{loc}^1$ and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is an orientation preserving C^1 -diffeomorphism of \mathbb{R} . Let $\psi \in C_0^\infty$ be a test function such that $\text{supp } \psi \subseteq (-R, R)$, $0 < R < \infty$. As $f \in H_{loc}^1$ there exists $(f_k)_{k \geq 1}$, $f_k \in C^\infty([\varphi(-R), \varphi(R)])$ such that $f_k \rightarrow f$ ($k \rightarrow \infty$) in $H^1((\varphi(-R), \varphi(R)))$. Then,

$$\begin{aligned} \langle (f \circ \varphi)', \psi \rangle &:= - \int_{-\infty}^{\infty} f(\varphi(x)) \cdot \psi'(x) dx = - \int_{\varphi(-R)}^{\varphi(R)} f(y) \cdot \frac{\psi'(\varphi^{-1}(y))}{\varphi'(\varphi^{-1}(y))} dy \\ &= - \lim_{k \rightarrow \infty} \int_{\varphi(-R)}^{\varphi(R)} f_k(y) \cdot \frac{\psi'(\varphi^{-1}(y))}{\varphi'(\varphi^{-1}(y))} dy \end{aligned} \quad (97)$$

and

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{-R}^R (f_k(\varphi(x)))' \cdot \psi(x) dx &= \lim_{k \rightarrow \infty} \int_{-R}^R f_k'(\varphi(x)) \cdot \varphi'(x) \cdot \psi(x) dx \\ &= \lim_{k \rightarrow \infty} \int_{\varphi(-R)}^{\varphi(R)} f_k'(y) \cdot \varphi'(\varphi^{-1}(y)) \cdot \frac{\psi(\varphi^{-1}(y))}{\varphi'(\varphi^{-1}(y))} dy \\ &= \int_{\varphi(-R)}^{\varphi(R)} f'(y) \cdot \varphi'(\varphi^{-1}(y)) \cdot \frac{\psi(\varphi^{-1}(y))}{\varphi'(\varphi^{-1}(y))} dy \\ &= \int_{-\infty}^{\infty} f'(\varphi(x)) \cdot \varphi'(x) \cdot \psi(x) dx \\ &= \langle f' \circ \varphi \cdot \varphi', \psi \rangle. \end{aligned} \quad (98)$$

Combining (97) with (98) we see that

$$(f \circ \varphi)' = f' \circ \varphi \cdot \varphi'. \quad (99)$$

Using a change of variables one easily sees that $f \circ \varphi$ and $f' \circ \varphi$ are locally square integrable. This and (99) then imply that $f \circ \varphi \in H_{loc}^1$. The proof of the last statement of (i) follows from (99) in a similar way. \square

Lemma 6.3. *Let $m \geq 1$ and $N \in \mathbb{R}$. Then there exists $C > 0$ such that for any $f \in W_N^m$ and for any $0 \leq j \leq m-1$,*

$$\sup_{x \in \mathbb{R}} |f^{(j)}(x)| \langle x \rangle^{N+j+\frac{1}{2}} \leq C \|f\|_{W_N^m}$$

and

$$\lim_{|x| \rightarrow \infty} |f^{(j)}(x)| \langle x \rangle^{N+j+\frac{1}{2}} = 0.$$

If $N \geq 0$ one has the continuous inclusions $W_N^m \subseteq H^m \subseteq C^{m-1}$.

Proof. First assume that $f \in W_N^1$ and $N \in \mathbb{R}$. Consider the function

$$g(x) := f(x)^2 \langle x \rangle^{2N+1}.$$

A direct differentiation shows that

$$|g'| \leq (2|N| + 1)(\langle x \rangle^N |f|)^2 + 2(\langle x \rangle^N |f|)(\langle x \rangle^{N+1} |f'|) \in L^1. \quad (100)$$

As $g' \in L^1$, $g(x) = c_- + \int_{-\infty}^x g'(y) dy = c_+ - \int_x^\infty g'(y) dy$ where $c_\pm \geq 0$ are non-negative constants. In particular,

$$\lim_{x \rightarrow \pm\infty} g(x) = c_\pm \geq 0.$$

Now, assume that $c_+ > 0$. Then, there exist $\alpha \geq 0$ and $\varepsilon > 0$ such that $g(x) \geq \varepsilon^2 > 0$ for any $x \geq \alpha$. In particular, for any $x \geq \alpha$,

$$|f(x)| \langle x \rangle^N > \frac{\varepsilon}{\langle x \rangle^{1/2}},$$

that contradicts $\langle x \rangle^N f \in L^2$. Hence, $c_+ = 0$. Arguing in a similar way we see that $c_- = 0$. In particular,

$$g(x) = \int_{-\infty}^x g'(y) dy = - \int_x^\infty g'(y) dy \quad (101)$$

and

$$\lim_{|x| \rightarrow \infty} g(x) = 0.$$

This shows that

$$\lim_{|x| \rightarrow \infty} |f(x)| \langle x \rangle^{N+\frac{1}{2}} = 0. \quad (102)$$

It follows from (100) and (101) that

$$|g(x)| \leq \int_{-\infty}^\infty |g'(y)| dy \leq (2|N| + 3) \|f\|_{W_N^1}.$$

Hence, there exists $C_N > 0$ such that for any $f \in W_N^1$ and for any $x \in \mathbb{R}$,

$$\langle x \rangle^{N+\frac{1}{2}} |f(x)| \leq C_N \|f\|_{W_N^1}. \quad (103)$$

Now, assume that $f \in W_N^m$, $m \geq 1$, and $N \in \mathbb{R}$. Take $0 \leq j \leq m-1$. Then

$$f^{(j)} \in W_{N+j}^{m-j} \subseteq W_{N+j}^1.$$

In view of (103) we have that for any $x \in \mathbb{R}$,

$$\langle x \rangle^{N+j+\frac{1}{2}} |f^{(j)}| \leq C_{N+j} \|f^{(j)}\|_{W_{N+j}^1} \leq C_{N+j} \|f\|_{W_N^m}.$$

This prove the lemma with $C := \max_{0 \leq j \leq m-1} C_{N+j}$. In view of (102) we also get that $\lim_{|x| \rightarrow \infty} |f^{(j)}(x)| \langle x \rangle^{N+j+\frac{1}{2}} = 0$. The last statement of the lemma follows from Sobolev's embedding theorem. \square

Arguing in a similar way one proves.

Lemma 6.4. *Let $m \geq 1$ and $N \in \mathbb{R}$. Then there exists $C > 0$ such that for any $f \in H_N^m$ and for any $0 \leq j \leq m-1$,*

$$\sup_{x \in \mathbb{R}} |f^{(j)}(x)| \langle x \rangle^N \leq C \|f\|_{H_N^m}$$

and

$$\lim_{|x| \rightarrow \infty} |f^{(j)}(x)| \langle x \rangle^N = 0.$$

If $N \geq 0$ one has the continuous inclusions $H_N^m \subseteq H^m \subseteq C^{m-1}$.

Proof. The proof of this statement follows the arguments of the proof of Lemma 6.3 applied to the function $g(x) := f(x)^2 \langle x \rangle^{2N}$. \square

The following Lemma can be proved in a straightforward way.

Lemma 6.5. *Assume that $m_1, m_2 \geq 1$, $N_1, N_2 \geq 0$, and $k \geq 0$. Then the mappings,*

$$\begin{aligned} (f, g) &\mapsto f \cdot g, & W_{N_1}^{m_1} \times W_{N_2}^{m_2} &\rightarrow W_{N_1+N_2}^{\min(m_1, m_2)}, \\ f &\mapsto f \cdot \frac{1}{\langle x \rangle^k}, & W_{N_1}^{m_1} &\rightarrow W_{N_1+k}^{m_1}, \\ f &\mapsto f \cdot \frac{x}{\langle x \rangle^{k+1}}, & W_{N_1}^{m_1} &\rightarrow W_{N_1+k}^{m_1}, \end{aligned}$$

and for $m_1 \geq 2$,

$$f \mapsto f', \quad W_{N_1}^{m_1} \rightarrow W_{N_1+1}^{m_1-1},$$

are continuous. In particular, for any $m \geq 1$, $N \geq 0$, W_N^m is a ring with continuous (pointwise) product.

As a consequence from Lemma 6.5 we get

Lemma 6.6. *Assume that $m_1, m_2 \geq 1$, $N_1, N_2, n_1, n_2, k \geq 0$. Then the mappings,*

$$\begin{aligned} (f, g) &\mapsto f \cdot g, & \mathcal{A}_{n_1, N_1}^{m_1} \times \mathcal{A}_{n_2, N_2}^{m_2} &\rightarrow \mathcal{A}_{n_1+n_2, \min(N_1+n_2, N_2+n_1)}^{\min(m_1, m_2)}, \\ f &\mapsto f \cdot \frac{1}{\langle x \rangle^k}, & \mathcal{A}_{n_1, N_1}^{m_1} &\rightarrow \mathcal{A}_{n_1+k, N_1+k}^{m_1}, \\ f &\mapsto f \cdot \frac{x}{\langle x \rangle^{k+1}}, & \mathcal{A}_{n_1, N_1}^{m_1} &\rightarrow \mathcal{A}_{n_1+k, N_1+k}^{m_1}, \end{aligned}$$

and for $m_1 \geq 2$,

$$f \mapsto f', \quad \mathcal{A}_{n_1, N_1}^{m_1} \rightarrow \mathcal{A}_{n_1+1, N_1+1}^{m_1-1},$$

are continuous. In particular, for any $m \geq 1$, $N \geq 0$, and $n \geq 0$, $\mathcal{A}_{n, N}^m$ is a ring with continuous (pointwise) product.

Remark 6.1. *As the mappings considered in Lemma 6.5 and Lemma 6.6 are multilinear and continuous they are C^∞ -smooth.*

Similar arguments imply the following analogs of Lemma 6.5 and Lemma 6.6.

Lemma 6.7. Assume that $m_1, m_2 \geq 1$, $N_1, N_2 \geq 0$, and $k \geq 0$. Then the mappings,

$$\begin{aligned} (f, g) &\mapsto f \cdot g, & H_{N_1}^{m_1} \times H_{N_2}^{m_2} &\rightarrow H_{N_1+N_2}^{\min(m_1, m_2)}, \\ f &\mapsto f \cdot \frac{1}{\langle x \rangle^k}, & H_{N_1}^{m_1} &\rightarrow H_{N_1+k}^{m_1}, \\ f &\mapsto f \cdot \frac{x}{\langle x \rangle^{k+1}}, & H_{N_1}^{m_1} &\rightarrow H_{N_1+k}^{m_1}, \end{aligned}$$

and for $m_1 \geq 2$,

$$f \mapsto f', \quad H_{N_1}^{m_1} \rightarrow H_{N_1}^{m_1-1},$$

are continuous. In particular, for any $m \geq 1$, $N \geq 0$, H_N^m is a ring with continuous (pointwise) product.

Lemma 6.8. Assume that $m_1, m_2 \geq 1$, $N_1, N_2, n_1, n_2, k \geq 0$. Then the mappings,

$$\begin{aligned} (f, g) &\mapsto f \cdot g, & \mathbb{A}_{n_1, N_1}^{m_1} \times \mathbb{A}_{n_2, N_2}^{m_2} &\rightarrow \mathbb{A}_{n_1+n_2, \min(N_1+n_2, N_2+n_1)}^{\min(m_1, m_2)}, \\ f &\mapsto f \cdot \frac{1}{\langle x \rangle^k}, & \mathbb{A}_{n_1, N_1}^{m_1} &\rightarrow \mathbb{A}_{n_1+k, N_1+k}^{m_1}, \\ f &\mapsto f \cdot \frac{x}{\langle x \rangle^{k+1}}, & \mathbb{A}_{n_1, N_1}^{m_1} &\rightarrow \mathbb{A}_{n_1+k, N_1+k}^{m_1}, \end{aligned}$$

and for $m_1 \geq 2$,

$$f \mapsto f', \quad \mathbb{A}_{n_1, N_1}^{m_1} \rightarrow \mathbb{A}_{n_1+1, N_1}^{m_1-1},$$

are continuous. In particular, for any $m \geq 1$, $N \geq 0$, and $n \geq 0$, $\mathbb{A}_{n, N}^m$ is a ring with continuous (pointwise) product.

Left composition with analytic maps: First we introduce some additional notation. Take $r \geq 0$ and denote

$$\mathbb{R}_r := \{x \in \mathbb{R} \mid |x| \geq r\}.$$

For any $m \geq 1$, $N \geq 0$, and $n \geq 1$, consider the weighted Sobolev space

$$W_N^m(\mathbb{R}_r) := \{f \in H_{loc}^m(\mathbb{R}_r) \mid \langle x \rangle^N f, \dots, \langle x \rangle^{N+m} f^{(m)} \in L^2(\mathbb{R}_r)\}$$

supplied with the norm

$$\|f\|_{W_N^m(\mathbb{R}_r)} := \left(\sum_{j=0}^m \int_{\mathbb{R}_r} |\langle x \rangle^{N+j} f^{(j)}(x)|^2 dx \right)^{1/2}$$

as well as the modified asymptotic space

$$\mathcal{A}_{n, N}^m(\mathbb{R}_r) := \left\{ u = \sum_{k=n}^N \left(a_k \frac{1}{\langle x \rangle^k} + b_k \frac{x}{\langle x \rangle^{k+1}} \right) + f \mid f \in W_N^m(\mathbb{R}_r) \right\}$$

supplied with the *weighted* norm

$$\|u\|_{\mathcal{A}_{n, N}^m(\mathbb{R}_r)} := \sum_{k=n}^N (|a_k| + |b_k|) / \langle r \rangle^k + \|f\|_{W_N^m(\mathbb{R}_r)}. \quad (104)$$

Note that if $r = 0$ then $\mathcal{A}_{n,N}^m(\mathbb{R}_r) \equiv \mathcal{A}_{n,N}^m$ and the corresponding norms coincide. It is also clear that for any given $r \geq 0$ the restriction map,

$$\mathcal{A}_{n,N}^m \rightarrow \mathcal{A}_{n,N}^m(\mathbb{R}_r), \quad u \mapsto u|_{\mathbb{R}_r},$$

is continuous and⁶

$$\|u\|_{\mathcal{A}_{n,N}^m(\mathbb{R}_r)} \rightarrow 0, \quad r \rightarrow \infty.$$

Lemma 6.9. *Assume that $m \geq 1$, $N \geq 0$, and $n \geq 1$. Then the following two statements hold*

- 1) $u \in \mathcal{A}_{n,N}^m$ if and only if $u \in H_{loc}^m((-r+1, r+1))$ and $u|_{\mathbb{R}_r} \in \mathcal{A}_{n,N}^m(\mathbb{R}_r)$.
- 2) *There exists a positive constant $C > 0$ so that for any $r \geq 0$, for any integer $p \geq 0$, and for any $u \in \mathcal{A}_{n,N}^m(\mathbb{R}_r)$,*

$$\|u^p u^{N+1}\|_{\mathcal{A}_{n,N}^m(\mathbb{R}_r)} \leq (C \|u\|_{\mathcal{A}_{n,N}^m(\mathbb{R}_r)})^p \|u^{N+1}\|_{\mathcal{A}_{n,N}^m(\mathbb{R}_r)}. \quad (105)$$

Proof. The proof of 1) follows directly from the definition of the spaces involved. Let us prove 2). Take $u \in \mathcal{A}_{n,N}^m(\mathbb{R}_r)$ and $g \in W_N^m(\mathbb{R}_r) \subseteq \mathcal{A}_{n,N}^m$. Then $u = \sum_{k=n}^N (a_k \frac{1}{\langle x \rangle^k} + b_k \frac{x}{\langle x \rangle^{k+1}}) + f$, $f \in W_N^m(\mathbb{R}_r)$. For any $n \leq k \leq N$ the pointwise product $(a_k / \langle x \rangle^k) \cdot g \in W_{N+k}^m \subseteq W_N^m$ (Lemma 6.5) and

$$\begin{aligned} \|(a_k / \langle x \rangle^k) \cdot g\|_{\mathcal{A}_{n,N}^m(\mathbb{R}_r)}^2 &= \sum_{j=0}^m \int_{\mathbb{R}_r} \left| \left(\frac{a_k}{\langle x \rangle^k} g(x) \right)^{(j)} \langle x \rangle^{N+j} \right|^2 dx \\ &\leq C_1^2 |a_k|^2 \sum_{j=0}^m \int_{\mathbb{R}_r} \left(\sum_{l=0}^j \frac{|g^{(l)}(x)|}{\langle x \rangle^k} \langle x \rangle^{N+l} \right)^2 dx \\ &\leq C_1^2 \frac{|a_k|^2}{\langle r \rangle^{2k}} \sum_{j=0}^m \int_{\mathbb{R}_r} \left(\sum_{l=0}^j |g^{(l)}(x)| \langle x \rangle^{N+l} \right)^2 dx \\ &\leq C_2^2 \|(a_k / \langle x \rangle^k)\|_{\mathcal{A}_{n,N}^m(\mathbb{R}_r)}^2 \|g\|_{\mathcal{A}_{n,N}^m(\mathbb{R}_r)}^2 \end{aligned} \quad (106)$$

where the positive constants C_1 and C_2 are independent of the choice of $a_k \in \mathbb{R}$ and $g \in W_N^m(\mathbb{R}_r)$. Similar arguments show that there exists $C_3 > 0$ such that for any $b_k \in \mathbb{R}$ and $g \in W_N^m(\mathbb{R}_r)$,

$$\|(b_k x / \langle x \rangle^{k+1}) \cdot g\|_{\mathcal{A}_{n,N}^m(\mathbb{R}_r)} \leq C_3 \|(b_k x / \langle x \rangle^{k+1})\|_{\mathcal{A}_{n,N}^m(\mathbb{R}_r)} \|g\|_{\mathcal{A}_{n,N}^m(\mathbb{R}_r)}. \quad (107)$$

By the product rule one also sees that there exists $C_4 > 0$ such that for any $f, g \in W_N^m$,

$$\|fg\|_{\mathcal{A}_{n,N}^m(\mathbb{R}_r)} \leq C_4 \|f\|_{\mathcal{A}_{n,N}^m(\mathbb{R}_r)} \|g\|_{\mathcal{A}_{n,N}^m(\mathbb{R}_r)}. \quad (108)$$

Combining (106), (107), and (108), we get that there exists a positive constant $C > 0$ such that for any $u \in \mathcal{A}_{n,N}^m(\mathbb{R}_r)$ and $g \in W_N^m(\mathbb{R}_r)$,

$$\|ug\|_{\mathcal{A}_{n,N}^m(\mathbb{R}_r)} \leq C \|u\|_{\mathcal{A}_{n,N}^m(\mathbb{R}_r)} \|g\|_{\mathcal{A}_{n,N}^m(\mathbb{R}_r)}. \quad (109)$$

⁶For simplicity of notation when estimating the $\mathcal{A}_{n,N}^m(\mathbb{R}_r)$ -norm of $u|_{\mathcal{A}_{n,N}^m(\mathbb{R}_r)}$ we write u instead of $u|_{\mathcal{A}_{n,N}^m(\mathbb{R}_r)}$.

As $n \geq 1$, the product $u^{N+1} \in W_N^m(\mathbb{R}_r)$ (Lemma 6.5) and the claimed inequality (105) then follows from (109) and an induction argument. \square

As a corollary we obtain the following

Proposition 6.1. *Assume that $m \geq 1$, $N \geq 0$, and $n \geq 1$. Let $\alpha : I \rightarrow \mathbb{R}$ be a real-analytic function in the open interval $I \subseteq \mathbb{R}$ that contains zero, $0 \in I$. Then, if $u \in \mathcal{A}_{n,N}^m$ and $\text{Image}(u) \subseteq I$ then $\alpha \circ u - \alpha(0) \in \mathcal{A}_{n,N}^m$.⁷ The same statement holds if \mathcal{A} is replaced by \mathbb{A} .*

Proof. Take $u \in \mathcal{A}_{n,N}^m$ so that $\text{Image}(u) \subseteq I$. As $0 \in I$ and as $\alpha : I \rightarrow \mathbb{R}$ is real-analytic there exists $\delta > 0$ such that

$$\alpha(x) = \sum_{j=0}^{\infty} \alpha_j x^j$$

where the series converges absolutely on $(-\delta, \delta)$. As $n \geq 1$ we see from Lemma 6.3 that there exists $r \geq 0$ so that for any $x \in \mathbb{R}_r$,

$$|u(x)| < \delta.$$

In particular, for any $x \in \mathbb{R}_r$,

$$(\alpha \circ u)(x) = \sum_{j=0}^{\infty} \alpha_j u(x)^j,$$

where the series converges absolutely. For any $x \in \mathbb{R}_r$, we have

$$(\alpha \circ u)(x) - \alpha_0 = \sum_{j=1}^N \alpha_j u(x)^j + \sum_{p=0}^{\infty} \alpha_{N+1+p} u(x)^p u(x)^{N+1}. \quad (110)$$

In view of Lemma 6.9,

$$\left\| \sum_{p=0}^{\infty} \alpha_{N+1+p} u^p u^{N+1} \right\|_{\mathcal{A}_{n,N}^m(\mathbb{R}_r)} \leq \sum_{p=0}^{\infty} |\alpha_{N+1+p}| (C \|u\|_{\mathcal{A}_{n,N}^m(\mathbb{R}_r)})^p \|u^{N+1}\|_{\mathcal{A}_{n,N}^m(\mathbb{R}_r)}. \quad (111)$$

By taking $r \geq 0$ so that

$$\|u\|_{\mathcal{A}_{n,N}^m(\mathbb{R}_r)} \leq \delta/C$$

we obtain that (110) converges in $\mathcal{A}_{n,N}^m(\mathbb{R}_r)$. Hence,

$$(\alpha \circ u - \alpha(0))|_{\mathbb{R}_r} \in \mathcal{A}_{n,N}^m(\mathbb{R}_r).$$

As $u \in H_{loc}^m(-(r+1), r+1)$, $\text{Image}(u) \subseteq I$, and as α is real-analytic in I , one concludes by standard arguments that

$$\alpha \circ u \in H_{loc}^m(-(r+1), r+1).$$

The statement of the Proposition now follows from Lemma 6.9, 1). The case of the space $\mathbb{A}_{n,N}^m$ is treated in the same way. \square

⁷ $\text{Image}(u) := \{u(x) \mid x \in \mathbb{R}\}.$

We will use Proposition 6.1 for proving the following important Lemmas.

Lemma 6.10. *Assume that $\psi := 1 + f > 0$ where $f \in \mathcal{A}_{n,N}^m$, $m \geq 1$, $N \geq 0$, and $n \geq 1$. Then $\frac{1}{\psi} - 1 \in \mathcal{A}_{n,N}^m$ and there exists an open neighborhood \mathcal{U} of zero in $\mathcal{A}_{n,N}^m$ such that for any $g \in \mathcal{U}$, $\psi + g > 0$, and the mapping,*

$$g \mapsto \frac{1}{\psi + g} - 1, \quad \mathcal{U} \rightarrow \mathcal{A}_{n,N}^m,$$

is real-analytic. The same statements are true if \mathcal{A} is replaced by \mathbb{A} .

Proof. The fact that $\frac{1}{\psi} - 1 \in \mathcal{A}_{n,N}^m$ follows from Proposition 6.1 applied to the real-analytic function $\alpha(x) := 1/(1+x)$, $\alpha : (-1, \infty) \rightarrow \mathbb{R}$. Let us prove the second statement of the Lemma. It follows from the continuity of the pointwise product in $\mathcal{A}_{n,N}^m$ and the continuity of the inclusion $\mathcal{A}_{n,N}^m \subseteq L^\infty$ that there exists an open neighborhood \mathcal{U} of zero in $\mathcal{A}_{n,N}^m$ such that

$$\left\| g \cdot \frac{1}{\psi} \right\|_{\mathcal{A}_{n,N}^m} < 1 \quad \text{and} \quad \left\| g \cdot \frac{1}{\psi} \right\|_{L^\infty} < 1.$$

Then for any $g \in \mathcal{U}$,

$$\frac{1}{\psi + g} = \frac{1}{\psi} \cdot \frac{1}{1 + g \cdot \frac{1}{\psi}} = \frac{1}{\psi} \left(1 + \sum_{j=1}^{\infty} (-1)^j \left(g \cdot \frac{1}{\psi} \right)^j \right)$$

where the series converges in $\mathcal{A}_{n,N}^m$. The case of the space $\mathbb{A}_{n,N}^m$ is treated in the same way. \square

Similar arguments show that the following Lemma hold.

Lemma 6.11. *Assume that $m \geq 2$, $N \geq 0$, and $n \geq 0$. Then for any $\varphi \in \mathcal{AD}_{n,N}^m$*

$$\frac{1}{\langle \cdot \rangle} \circ \varphi = \frac{1}{\langle x \rangle} (1 + w), \quad w \in \mathcal{A}_{n+1,N+1}^m$$

where $\frac{1}{\langle \cdot \rangle} \circ \varphi$ stands for $1/\sqrt{1 + \varphi(x)^2}$. Moreover, the mapping,

$$\varphi \mapsto \frac{1}{\langle \cdot \rangle} \circ \varphi - \frac{1}{\langle x \rangle}, \quad \mathcal{AD}_{n,N}^m \rightarrow \mathcal{A}_{n+2,N+2}^m$$

is real-analytic. The same statements are true if \mathcal{A} is replaced by \mathbb{A} .

Proof. Take $\varphi \in \mathcal{AD}_{n,N}^m$. Then $\varphi(x) = x + u(x)$, $u \in \mathcal{A}_{n,N}^m$, and

$$\frac{1}{\langle \cdot \rangle} \circ \varphi = \frac{1}{\sqrt{1 + (x + u)^2}} = \frac{1}{\langle x \rangle} \cdot \frac{1}{\sqrt{1 + \left(2 \frac{x}{\langle x \rangle} \frac{u}{\langle x \rangle} + \left(\frac{u}{\langle x \rangle} \right)^2 \right)}}. \quad (112)$$

In view of Lemma 6.6, the expression $2\frac{x}{\langle x \rangle}\frac{u}{\langle x \rangle} + \left(\frac{u}{\langle x \rangle}\right)^2 \in \mathcal{A}_{n+1, N+1}^m$. As $1 + \left(2\frac{x}{\langle x \rangle}\frac{u}{\langle x \rangle} + \left(\frac{u}{\langle x \rangle}\right)^2\right) > 0$ we obtain from Proposition 6.1, applied to the real-analytic function $\alpha(x) := 1/\sqrt{1+x}$, $\alpha : (-1, \infty) \rightarrow \mathbb{R}$, that

$$\frac{1}{\sqrt{1 + \left(2\frac{x}{\langle x \rangle}\frac{u}{\langle x \rangle} + \left(\frac{u}{\langle x \rangle}\right)^2\right)}} - 1 \in \mathcal{A}_{n+1, N+1}^m.$$

This together with (112) completes the proof of the first statement of the Lemma. The second statement follows as in the proof of the second statement of Lemma 6.10. The case of the space $\mathbb{A}_{n, N}^m$ is treated in the same way. \square

References

- [1] V. Arnold, *Sur la geometrie differentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluids parfaits*, Ann. Inst. Fourier, **16**, 1(1966), 319-361
- [2] I. Bondareva, M. Shubin, *Growing asymptotic solutions of the Korteweg-de Vries equation and of its higher analogues*, Dokl. Akad. Nauk SSSR, **267**(1982), no. 5, 1035-1038
- [3] I. Bondareva, M. Shubin, *Uniqueness of the solution of the Cauchy problem for the Korteweg-de Vries equation in classes of increasing functions*, Vestnik Moskov. Univ. Ser. I Mat. Mekh, 1985, no. 3, 35-38
- [4] I. Bondareva, M. Shubin, *Equations of Korteweg-de Vries type in classes of increasing functions*, J. Soviet Math., **51**(1990), no. 3, 2323-2332
- [5] R. Camassa, D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett, **71**(1993), 1661-1664
- [6] A. Constantin, *Existence of permanent and breaking waves for a shallow water equation: a geometric approach*, Ann. Inst. Fourier, Grenoble, **50**(2000), no. 2, 321-362
- [7] A. Constantin, J. Escher, *Global existence and blow-up for a shallow water equation*, Annali Sc. Norm. Sup. Pisa, **26**(1998), 303-328
- [8] A. Constantin, J. Escher, *On the blow-up rate and the blow-up set of breaking waves for a shallow water equation*, Math. Z., **233**(2000), 75-91
- [9] A. Constantin, J. Escher, *Global weak solutions for a shallow water equation*, Indiana Univ. Math. J., **47**(1998), no. 4, 1527-1545
- [10] C. De Lellis, T. Kappeler, P. Topalov, *Low regularity solutions of the Camassa-Holm equation*, Comm. Partial Differential Equations, **32**(2007), no. 1-3, 87-126

- [11] D. Ebin, J. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. Math., **92**(1970), 102-163
- [12] A. Fokas, B. Fuchssteiner, *Symplectic structures, their Bäcklund transformation and hereditary symmetries*, Physica D, **4**(1981), 47-66
- [13] D. Holm, M. Staley, *Wave structure and nonlinear balances in a family of evolutionary PDEs*, SIAM J. Applied Dynamical Systems, **2**(2003), no. 3, 323-380
- [14] D. Holm, M. Staley, *Nonlinear balance and exchange of stability in dynamics of solitons, peakons, ramps/cliffs and leftons in $1+1$ nonlinear PDE*, Phys. Lett. A, **308**(2003), 437-444
- [15] H. Inci, T. Kappeler, P. Topalov, *On the regularity of the composition of diffeomorphisms*, Mem. Amer. Math. Soc., **226**(2013), no. 1062
- [16] T. Kappeler, P. Perry, M. Shubin, P. Topalov, *Solutions of mKdV in classes of functions unbounded at infinity*, J. Geom. Anal., **18**(2008), no. 2, 443-477
- [17] S. Lang, *Differential manifolds*, Addison-Wesley Series in Mathematics, 1972
- [18] R. McOwen, P. Topalov, *Groups of asymptotic diffeomorphisms*, preprint
- [19] G. Misiolek, *A shallow water equation as a geodesic flow on the Bott-Virasoro group*, J. Geom. Phys., **24**(1998), 203-208
- [20] G. Misiolek, *Classical solutions of the periodic Camassa-Holm equation*, GAFA, **12**(2002), 1080-1104
- [21] V. Ovsienko, B. Khesin, *Korteweg-de Vries superequations as an Euler equation*, Functional Anal. Appl., **21**(1987), 81-82